Department of Physics

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MSc Dissertation

Instantons in Quantum Field Theory and String Theory

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Contents

1 Introduction and outline

It is a generic feature of many areas of modern physics that the fundamental equations underlying the theory have been found. Nevertheless, solving those equations analytically is usually impossible for all but the most simple cases. In practice, numerical methods or approximation schemes have therefore to be employed.
An example of such a scenario is perturbation theory in quantum mechanics and quantum field theory: One begins with a free system whose exact solution is known and then calculates perturbative corrections order by order in some coupling parameter of the interaction. Despite the many successes of this approach, it turns out that, maybe unsurprisingly, some properties of the system cannot be described perturbatively. In some cases, the inclusion of non-perturbative effects can alter the qualitative behavior of a theory drastically, therefore justifying their detailed study.

1.1 A simple quantum mechanical example

In order to illustrate the importance of non-perturbative effects in quantum theory, we for now restrict ourselves to quantum mechanics and the phenomenon of barrier penetration. As is known from standard courses of quantum mechanics, there is a non-zero probability for a particle to penetrate areas that would classically be forbidden. This effect is usually quantified using the WKB approximation and one obtains a transmission probability of

$$|T| = \exp\left(-\frac{1}{\hbar}\int_{x_1}^{x_2} \mathrm{d}x\sqrt{2(V-E)}\right) \ . \tag{1.1}$$

Note the dependence on $\hbar$: Doing perturbation theory in powers of $\hbar$ one would never have encountered this contribution. We will now follow Coleman's lucid discussion [10] in order to reproduce this result.

Consider a particle of unit mass sitting at the bottom of an even double-well potential $V(x)$ and let $V(x)$ be shifted such that $V(\pm a) = 0$ at its minima as illustrated in Figure

1. We are interested in the matrix element

$$\langle -a|\, \mathrm{e}^{-iHT/\hbar}\, |a\rangle = \sum_n \mathrm{e}^{-iE_nT/\hbar} \langle x_f|n\rangle\langle n|x_i\rangle \ , \tag{1.2}$$

where $H = \frac{p^2}{2} + V(x)$ is the usual Hamilton operator and $\{|n\rangle\}$ is the corresponding set of eigenstates.

Our approach will use the Feynman path integral to rewrite this expression as

$$\langle -a|\, \mathrm{e}^{-iHT/\hbar}\, |a\rangle = N \int [\mathrm{d}x] \exp\left(\frac{i}{\hbar}\tilde{S}[x(t)]\right) \ . \tag{1.3}$$

Here $\tilde{S}$ is the classical action, N is a normalization constant and by $\int[\mathrm{d}x]$ we mean functional integration over all paths $x(t)$ satisfying $x(-T/2) = -a$ and $x(T/2) = a$. For convenience, let us now perform a Wick rotation to imaginary time by replacing $t \to -it$. Doing so gives

$$\langle -a|\, \mathrm{e}^{-HT/\hbar}\, |a\rangle = N \int [\mathrm{d}x] \exp\left(-\frac{1}{\hbar}S[x(t)]\right) \ , \tag{1.4}$$

where the Wick rotation has the effect of inverting the sign of the potential, so that the Euclidean action reads

$$S = \frac{p^2}{2} + V(x) \ . \tag{1.5}$$

Let us try to perform the path integral. For a generic potential $V(x)$, this cannot be done and one is therefore led to approximate $S[x]$ by expanding it up to quadratic order around its stationary points:

$$\begin{aligned} S[x(t)] &\approx S[\bar{x}(t)] + \int \mathrm{d}t_1\, \delta x(t_1) \frac{\delta S}{\delta x(t_1)} + \frac{1}{2}\int \mathrm{d}t_1\, \mathrm{d}t_2\, \delta x(t_1) \frac{\delta^2 S}{\delta x(t_1)\delta x(t_2)} \delta x(t_2) \\ &= S[\bar{x}(t)] + \frac{1}{2}\int \mathrm{d}t_1 \mathrm{d}t_2\, \delta x(t_1)\delta(t_1 - t_2)\left(-\frac{\partial^2}{\partial t_2^2} + V''(x)\right)\delta x(t_2) \end{aligned} \tag{1.6}$$

Note that the paths $\bar{x}(t)$ are precisely the solutions to the classical motion in the inverted

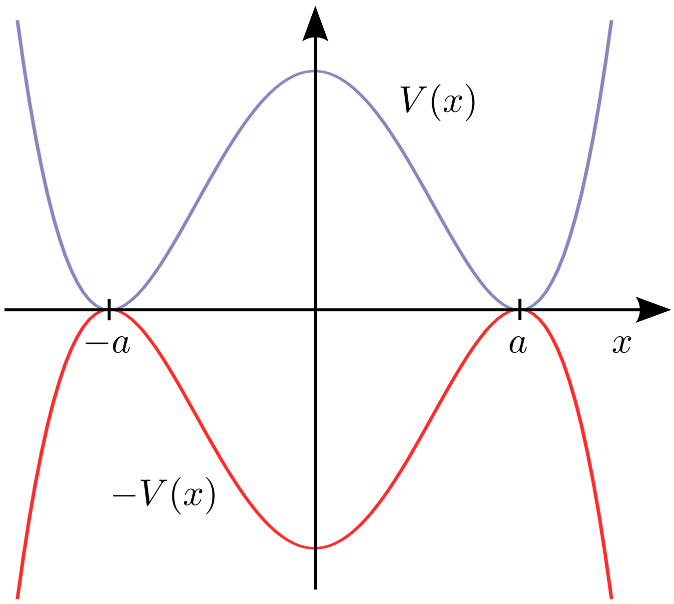

Figure 1: Example of a double well potential $V(x)$ in blue and its inversion below

potential $-V(x)$ with a total energy of

$$E = \frac{1}{2}\dot{\bar{x}}^2 - V(\bar{x}) \; . \tag{1.7}$$

Imagine a classical particle sitting at the bottom of one of the wells. Its energy will be close to zero and, integrating the above equation, its classical motion will therefore approximately satisfy

$$t - t_1 = \int_{-a}^{\bar{x}} \mathrm{d}x \sqrt{2V(x)} \; . \tag{1.8}$$

For $t - t_1 >> 1$ the particle's position will be close to a and one therefore has

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}t} \approx \sqrt{V''(a)(a-\bar{x})^2} = \omega(a - \bar{x}) \; , \tag{1.9}$$

where $V''(a) = \omega^2$. Hence

$$a - \bar{x} = \mathrm{e}^{-\omega(t-t_1)} \quad \text{and} \quad S_0 \equiv S[\bar{x}(t)] = \int_{-a}^{a} \mathrm{d}x \, \sqrt{2V(x)} \tag{1.10}$$

and we see that our solution corresponds to the classical particle rolling down the hill once, or, equivalently, the quantum mechanical particle penetrating the barrier once. Its motion is localized in time and has a size of about $1/\omega$. Having obtained this solution, we can produce arbitrarily many more by adding paths that correspond to the classical particle rolling back and forth n times. If we assume that the times

$$-T/2 < t_1 < \ldots < t_{2n+1} < T/2$$

at which this happens are well separated, then the solutions will not interfere and the resulting action will be roughly equal to $(2n+1) \cdot S_0$.

With this result at our hands, we can finally evaluate the path integral. In order to account for the different classical solutions, we sum over all natural numbers n and integrate over the times $t_1, \ldots, t_{2n+1}$ around which the motion is centered. The integral over the centers gives rise to a factor of

$$\int_{-T/2}^{T/2} \mathrm{d}t_1 \int_{-T/2}^{t_1} \mathrm{d}t_2 \ldots \int_{-T/2}^{t_{2n}} \mathrm{d}t_{2n+1} = \frac{T^{2n+1}}{(2n+1)!} \,. \tag{1.11}$$

One then calculates

$$\langle -a| \, \mathrm{e}^{-HT/\hbar} \, |a\rangle = \sum_{n=1}^{\infty} \frac{\left(\mathrm{e}^{-S_0} T\right)^{2n+1}}{(2n+1)!} N[\det\left(-\partial_t^2 + \omega^2\right)]^{-\frac{1}{2}} [1 + \mathcal{O}(\hbar)] \,. \tag{1.12}$$

Care must be taken to evaluate the functional determinant in the right background, but essentially the calculation boils down to arguing that for almost all times t the particle will be sitting at the bottom of one of the wells and that the correction coming from the movement of the particle only contributes by a multiplicative factor K^{2n+1}. For the "static" case the functional determinant can be evaluated to give $N[\det\left(-\partial_t^2 + \omega^2\right)]^{-\frac{1}{2}} \approx \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} \mathrm{e}^{-\omega T/2}$.
Putting it all together and performing the sum, one therefore arrives at

$$\langle -a| \, \mathrm{e}^{-HT/\hbar} \, |a\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} \mathrm{e}^{-\omega T/2} \frac{1}{2} \left[\exp\left(K \mathrm{e}^{-S_0/\hbar}\right) + \exp\left(-K \mathrm{e}^{-S_0/\hbar}\right)\right] \,. \tag{1.13}$$

Direct comparison of this expression with Eq. (1.2) shows that two eigenstates contribute, namely the odd and the even superpositions of harmonic oscillator ground states centered around the two different wells. Their energy differs by a factor proportional to $\mathrm{e}^{-S_0/\hbar}$, thereby reproducing Eq. (1.1). A more detailed discussion of this result including an evaluation of K is contained in [10], but for now we would like to summarize a few key points:

- A semi-classical approximation of the path integral leads to classical solutions localized in time. Because of that property, they were dubbed instantons by 't Hooft [39, 38].
- Instanton solutions can give non-perturbative contributions in $\hbar$.
- The instanton solutions are parametrized by a continuous parameter t_1, the time around which they are centered. This space of solutions is called the instanton moduli space. A more careful derivation shows that the parameter t_1 is related to a zero mode of $-\partial_t^2 + V''(x)$ which has to be removed before calculating the determinant. In order to evaluate the path integral one has to integrate over all possible parameters.
- We did not need the exact form of the path $\bar{x}(t)$, but only its action and the dependence on external parameters.

1.2 Outline of the dissertation

A natural generalization is now to apply the same reasoning to quantum field theory and search for classical solutions around which to perform the semi-classical approximation of the path integral. As a matter of fact, the solutions that we will encounter share many features with the toy example discussed in the previous section, but before we begin in earnest, let us give a short overview of the structure of this thesis:

- Chapter 2 is concerned solely with classical instanton solutions in various gauge theories. Compared to our simple toy model, the structure will be much richer and therefore deserves a more detailed treatment. Nevertheless, many of the features one encounters in doing so will be similar to what we have seen in the previous section. Among other things, the instanton moduli space will turn out to carry all the information we are interested in and will hence be the main object of our study.

We review the ADHM construction of instanton solutions, touch upon the geometric properties of the instanton moduli space and explore their connections to supersymmetric gauge theories appearing naturally in certain brane constructions of Type II string theory. As an application of this connection, we report on recent attempts to study instanton moduli spaces using methods developed in the context of supersymmetric gauge theories [7, 20].

- Having analyzed the classical properties of instantons, Chapter 3 clearly puts the emphasis on their implications for a quantum theory. To do so, we summarize some of the applications they have found in quantum field theory. Among other things, we describe the relation to anomalies and vacuum energy.

- Ordinarily, determining all instanton contributions in a quantum field theory would appear to be a hopeless endeavor. Yet progress has been made in the more controlled scenarios of supersymmetric gauge theories. Chapter 4 is therefore devoted to presenting the ingenious solution of the low-energy $\mathcal{N} = 2$ supersymmetric field theory by Seiberg and Witten [34], in which they exploited symmetry arguments in order to determine all instanton corrections without the help of the usual field-theoretical machinery.

- Last of all, we give a very brief sketch of Nekrasov's work [32] to apply localization techniques from topological field theory in order to reproduce the result of Seiberg and Witten. Unlike the rest of the dissertation, the discussion of Nekrasov's partition function will omit some calculations as they could easily occupy a review by themselves.

Before diving into the subject, let it be very clear that none of the results reviewed below are original work, but they only represent the author's attempt to understand some of the fascinating features arising non-perturbatively in quantum field theories. Instantons seem to have found many rich applications in the almost four decades after their first discovery [4] and we can therefore only scratch the surface of this topic. There also seem to be many relations to string theory, as well as pure topology and geometry which we will try to highlight. Nevertheless, this is only a small selection of topics and much of what is covered will only be superficial. For that we apologize in advance and we will try to give references to more complete reviews at the beginning of each chapter.

1.3 Acknowledgements

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During this year, we regularly had interesting and fun discussions in our class, but with respect to this thesis, I am especially grateful to Colin Rylands for discussions about Seiberg-Witten theory. Last, but certainly not least, I would like to thank Amihay Hanany for the numerous "lunch meetings" and the many occasions he gave me advise and support, both related to physics and not. I thoroughly enjoyed these times and I will certainly miss them.

2 Instantons and their moduli spaces

In this chapter we obtain classical instanton solutions to various gauge theories and examine the geometrical properties of the respective instanton moduli spaces. There are several good reviews of varying difficulty. David Tong's lecture notes [40] give a very intuitive introduction to the topic and emphasize the connections to supersymmetry and string theory. A much more thorough and fairly technical account of instanton calculations in various supersymmetric is provided by Dorey, Hollowood, Khoze, and Mattis [12].

2.1 Winding numbers and instanton equations

In the first chapter we hinted at some of the properties that can emerge when considering non-perturbative contributions to the path integral. Let us now try to generalize this concept by looking at quantum field theories and the corresponding path integral in Euclidean spacetime

$$\mathcal{Z}(J_m(x)) = \int \mathcal{D}A \, \exp\left(-S[A(x)] + \int \mathrm{d}^4x \, \mathrm{Tr}\, J_m(x) A^m(x)\right) \tag{2.1}$$

with an arbitrary source term J_m. We use Latin indices ranging from 1 to 4 in order to indicate that we are not in Minkowski space. Raising or lowering indices does not have any effect and we use the notation with one index downstairs and another upstairs purely to indicate that these indices are summed over.

In order to perform a semi-classical approximation, we would again expand the action up to quadratic powers in A_m, yielding

$$S[A_m(x)] = S[\bar{A}_m(x)] + \frac{1}{2}\int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \, \delta A_m(x_1) \frac{\delta^2 S}{\delta A_m(x_1)\delta A_n(x_2)} \delta A_n(x_2) + \dots \, , \tag{2.2}$$

where $\delta A_m(x) = A_m(x) - \bar{A}_m(x)$ and $\bar{A}_m(x)$ is a solution to the classical equations of motion with finite action.

For the time being we restrict ourselves to pure $SU(N)$ Yang-Mills gauge theories. We take our gauge fields $A_m(x)$ to be Anti-Hermitian and pick a basis of the adjoint representation of $SU(N)$ such that $A_m(x) = A^a_m(x)T^a$, $\mathrm{Tr}(T^aT^b) = -\frac{1}{2}\delta^{ab}$ and $[T^a, T^b] = f^{abc}T^c$. Absorbing the coupling constant g into the fields A_m, the covariant derivative in the fundamental representation reads $\mathcal{D}_m = \partial_m + A_m$ and $F_{mn} = [\mathcal{D}_m, \mathcal{D}_n] = \partial_\mu A_\nu - \partial_n A_m + [A_m, A_n]$.

Last of all, gauge transformations $U(x) \in SU(N)$ act on A_m as

$$A_m \to UA_mU^{-1} + U\partial_m U^{-1} \ . \tag{2.3}$$

With these choices the gauge theory action reads

$$S = -\frac{1}{2g^2}\int \mathrm{d}^4x \, \mathrm{Tr}\, F_{mn}F^{mn} \tag{2.4}$$

and the classical equations of motion are

$$\mathcal{D}^m F_{mn} = 0 \ . \tag{2.5}$$

Note that this implies that contributions to $\mathcal{Z}(J)$ arising from configurations with non-zero action S_0 are weighted by a factor of e^{-S_0}. Since their contribution contains inverse powers of g, it could never be reproduced by an ordinary perturbation expansion in powers of g.

2.1.1 Homotopy

Before we begin to derive actual instanton equations, one can make a very useful observation which helps to classify potential classical solutions to the equations of motion. Since we are solely interested in finite action solutions to Eq. (2.5), F_{mn} must vanish as we approach the boundary of spacetime $\mathbb{R}^4$. According to Eq. (2.3) this translates into the condition that

$$A_m \xrightarrow{r\to\infty} U\partial_m U^{-1} \quad \text{with} \quad U(x) \in SU(N) \tag{2.6}$$

and the restriction of any gauge field A_m to spatial infinity therefore represents a map from $\partial\mathbb{R}^4 = S^3$ to $SU(N)$. Fortunately, smooth maps from S^n to spaces $\mathcal{M}$ are classified according to their n^{th} homotopy class. This is a well-known mathematical result, but let us quickly recapitulate the basics:

Consider two maps f, g from S^1 to $\mathcal{M}$. We consider f and g to be equivalent if there is another map $F : [0,1] \times S^1 \to \mathcal{M}$ such that

$$F(0,t) = f(t) \quad \text{and} \quad F(1,t) = g(t) \quad \forall t \in S^1 \tag{2.7}$$

and F is smooth. f and g therefore share the same first homotopy class $[f] = [g] \in \pi_1(\mathcal{M})$ if they can smoothly be deformed into each other. The generalization to maps from S^n follows along analogous lines and the respective homotopy groups $\pi_n(\mathcal{M})$ are a topological property of $\mathcal{M}$. A more detailed discussion is for example contained in [29], but here we simply quote the relevant result:

$$\pi_3(SU(N)) \cong \mathbb{Z} \tag{2.8}$$

This implies that every classical solution carries an integer $k \in \mathbb{Z}$ called instanton charge and the instanton moduli space must therefore be partitioned into infinitely many disconnected subspaces

$$\mathfrak{M} = \bigoplus_{k=-\infty}^{k=\infty} \mathfrak{M}_k \ . \tag{2.9}$$

Intuitively, k can be imagined as counting the number of times the group $SU(N)$ is wrapped around spatial infinity S^3 and a further result from mathematics states that this is measured by the following integral:

$$k = \frac{1}{24\pi^2} \int_{S^3} \mathrm{d}^3 S_m \epsilon^{mnrs} \operatorname{Tr} \Big((\partial_n U^{-1}) U (\partial_r U^{-1}) U (\partial_s U^{-1}) U \Big) \tag{2.10}$$

Clearly, the ordinary gauge field background $A_m \equiv 0$ has zero instanton charge and zero action. But what about solutions with non-zero k? A simple argument, given e.g. in [40] goes as follows: Picture $\mathbb{R}^4$ as a cone over S^3. At the origin, A_m must be single-valued and therefore have homotopy class $[0]$. If $[A|_\infty] \neq [0]$, then at some point in space, A_m must stop being pure gauge. We would therefore expect its action to be non-zero.

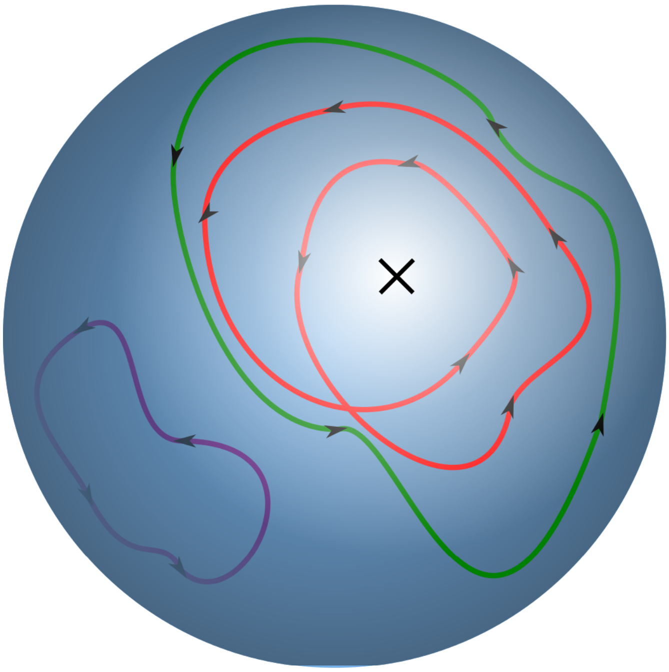

Figure 2: Paths of different homotopy classes on the punctured 2-sphere with homotopy group $\mathbb{Z}$.

2.1.2 Instanton equations

Having made the observation that the instanton charge is really a topological quantity, one can employ a trick first used in [4] to derive actual equations of motion by finding a lower bound for the gauge field action.

Exploiting the fact that the square to the dual field string tensor $\star F_{mn} = \frac{1}{2}\epsilon_{mnrs}F^{rs}$ is equal to the square of F_{mn}, we can rewrite S by completing the square:

$$\begin{aligned} S &= -\frac{1}{2g^2}\int \mathrm{d}^4x \,\mathrm{Tr}\, F_{mn}F^{mn} \\ &= -\frac{1}{4g^2}\int \mathrm{d}^4x \,\mathrm{Tr}\left((F_{mn} \mp \star F_{mn})^2 \pm 2F_{mn}\star F^{mn}\right) \\ &\geq \mp\frac{1}{2g^2}\int \mathrm{d}^4x \,\mathrm{Tr}\, F_{mn}\star F^{mn} \end{aligned}$$

The last integrand turns out to be a total derivative and can be rewritten as a surface integral over spatial infinity.

$$S \geq \mp\frac{1}{2g^2}\int \mathrm{d}^3S_m \,\epsilon^{mnrs}\,\mathrm{Tr}\left(A_nF_{rs} + \frac{2}{3}A_nA_rA_s\right) \tag{2.11}$$

Inserting the expression (2.6) for A_m at infinity, this again turns out to be proportional to Eq. (2.10) and S is therefore bounded by

$$S \geq \frac{8\pi^2}{g^2}|k|. \tag{2.12}$$

The inequality is saturated if and only if

$$F_{mn} = \text{sgn}(k) \star F_{mn} \, . \tag{2.13}$$

In fact, we have obtained more than just a lower bound for the action corresponding to a certain instanton charge. Solutions satisfying Eq. (2.13) will necessarily be minima of the action for a certain value of k and therefore be solutions to the classical equations of motion. Note further that parity maps solutions with instanton charge k to solutions having the opposite charge $-k$ (so-called anti-instantons), allowing us to concentrate on those with positive charge.

As a side remark, it should be noted that finding a clever way of completing the square turns out to be a handy tool not only for instantons. Indeed, it can be used to derive similar equations of motion for other topological defects such as monopoles or vortices.

2.1.3 An $SU(2)$ solution

Naturally, the next question one would now ask is whether solutions to Eq. (2.13) have been found, and indeed, they have been. Before writing down the actual solution, let us introduce some more notation. As will be seen shortly, instantons are most naturally described by quaternionic objects. It is convenient to exploit that $\mathfrak{so}(4) = \mathfrak{su}(2) \times \mathfrak{su}(2)$ which in turn implies that the covering group of $SO(4)$ is $SU(2)_L \times SU(2)_R$. Using an intertwiner between the vector representation $[1, 0]$ of $SO(4)$ and the $[1; 1]$ representation of $SU(2)_L \times SU(2)_R$ we can rewrite any spacetime vector as

$$x_{\alpha\dot{\alpha}} = x^m \sigma_{m\alpha\dot{\alpha}} \quad \text{or, equivalently,} \quad \bar{x}^{\dot{\alpha}\alpha} = x^m \bar{\sigma}_m^{\dot{\alpha}\alpha} \, , \tag{2.14}$$

where the respective $SU(2)_L$ and $SU(2)_R$ indices α and $\dot{\alpha}$ run from 1 to 2. Furthermore $\sigma_{n\alpha\dot{\alpha}} = (i\vec{\sigma}, \mathbb{1})$ and $\bar{\sigma}_n^{\dot{\alpha}\alpha} = (-i\vec{\sigma}, \mathbb{1})$ are the Euclidean four-dimensional version of the usual 2×2 Pauli matrices $\vec{\sigma}$. One can then define self-dual and anti-self-dual matrices

$$\sigma_{mn} = \frac{1}{4}\left(\sigma_m \bar{\sigma}_n - \sigma_n \bar{\sigma}_m\right) \quad \text{and} \quad \bar{\sigma}_{mn} = \frac{1}{4}\left(\bar{\sigma}_m \sigma_n - \bar{\sigma}_n \sigma_m\right) \tag{2.15}$$

satisfying

$$\sigma_{mn} = \frac{1}{2}\epsilon_{mnkl}\sigma^{kl} \quad \text{and} \quad \bar{\sigma}_{mn} = \frac{1}{2}\epsilon_{mnkl}\bar{\sigma}^{kl} \ . \tag{2.16}$$

We can now write down the solution for the simplest possible case of an $SU(2)$ instanton with charge $k = 1$. The so-called BPST instanton [4] reads

$$A_n = \frac{2\rho^2 (x-X)^m \bar{\sigma}_{mn}}{(x-X)^2\left((x-X)^2 + \rho^2\right)} \tag{2.17}$$

and has a field strength of

$$F_{mn} = \frac{4\rho^2 \sigma_{mn}}{\left((x-X)^2 + \rho^2\right)^2} \ . \tag{2.18}$$

There are several things to note about this solution:

- The solution is a co-dimension four object, with the suppression of the field strength behaving as $(x-X)^{-4}$ where X^m is the center of the instanton. It is therefore localized both at a point in space and an instant in time, which justifies its name.
- The fact that A_n looks singular around X^m is a gauge artifact as long as $\rho > 0$ and can be removed by choosing a different gauge, such as

  $$A_n = \frac{2(x-X)^m \sigma_{mn}}{(x-X)^2 + \rho^2} \tag{2.19}$$

  obtained by gauging with $U^{-1}(x) = \frac{(x-X)_m \bar{\sigma}^m}{|x-X|}$. For $\rho = 0$ there is an actual singularity as revealed by the field strength, which will be discussed later.
- Equation (2.17) possesses several parameters that one can choose freely, namely four real numbers X^m parametrizing the position of the instanton and another one,

ρ, controlling its size. In fact, one can also perform global gauge transformations such that the most general form for A_n reads

$$A_n = \frac{2\rho^2(x-X)^m U\bar{\sigma}_{mn}U^{-1}}{(x-X)^2\left((x-X)^2+\rho^2\right)} \quad \text{with} \quad U \in SU(2)\,. \tag{2.20}$$

Although these global gauge transformations leave the field strength invariant, they represent symmetries of the physical theory rather than redundancies of our description and their inclusion leads to richer mathematical structure. In total, there are eight real number called *generalized coordinates*. They parametrize $\mathfrak{M}$, the instanton moduli space, and are a generalization of the coordinate t_1 encountered in the introduction.

One could now ask how to extend this solution to more general cases. Having found the solution for $SU(2)$, it turns out to be easy to find solutions of the same instanton charge for higher-rank gauge groups $SU(N)$ simply by embedding the $SU(2)$ solution. The space of inequivalent embeddings is given by

$$\frac{SU(N)}{S\left(U(N-2)\times U(2)\right)}\,, \tag{2.21}$$

where $SU(N)$ acts by rotating the $SU(2)$ solution and the stabilizer $S\left(U(N-2)\times U(2)\right)$ acts either by rotating the complement of the embedded solution or by global gauge transformations already contained in Eq. (2.20). This coset space has dimension $N^2 - 1 - \left((N-2)^2 + 2^2 - 1\right) = 4N - 8$, adding up to a total of $4N$ coordinates. Later on, we will see that these are indeed all solutions, but before we do so, let us study the instanton moduli space in its own right.

2.2 General properties of $\mathfrak{M}$

Until now, all we have seen is that solutions to Eq. (2.5) with non-zero winding number have several parameters. In this section we explain that the instanton moduli space $\mathfrak{M}$ has an interesting geometrical structure that already contains much of the information we are interested in. We already know that $\mathfrak{M}$ is partitioned into subspaces $\mathfrak{M}_k$, but let us now ask what their geometry looks like: What is $\dim \mathfrak{M}_k$? How much structure does it admit? Is there a canonical metric? In order to answer these questions, one has to introduce a few more tools.

2.2.1 Zero modes

The most convenient way of defining the dimension of $\mathfrak{M}_k$ is by counting in how many inequivalent directions one can perturb $A^m \in \mathfrak{M}_k$ such that $A^m + \delta A^m$ is still a solution to the equations of motion. Linearizing Eq. (2.5) one finds that δA^m must satisfy

$$\mathcal{D}_m \delta A_n - \mathcal{D}_n \delta A_m = \epsilon_{mnrs} \mathcal{D}^r \delta A^s \ . \tag{2.22}$$

Naturally, any local gauge transformation $U(x) \in SU(N)$ will also be a solution to the linearized equations of motion and should be excluded. One therefore requires the zero mode δA_n to be orthogonal to all gauge transformations with respect to some inner product. We choose

$$0 = \langle \mathcal{D}^n \Omega, \delta A_n \rangle \equiv -2 \int \mathrm{d}^4 x \, \mathrm{Tr} \left(\mathcal{D}^n \Omega \delta A_n \right) \quad \forall \, \Omega(x) \in SU(N) \ . \tag{2.23}$$

Integration by parts then gives

$$\mathcal{D}^n \delta A_n = 0 \ . \tag{2.24}$$

Defining $\not{\mathcal{D}} = \mathcal{D}^n \sigma_n$ and $\bar{\not{\mathcal{D}}} = \mathcal{D}^n \bar{\sigma}_n$, (2.22) and (2.24) combine nicely into

$$\bar{\not{\mathcal{D}}}^{\dot{\alpha}\alpha} A_{\alpha\dot{\beta}} = 0 \ . \tag{2.25}$$

If we write our zero mode as

$$\delta_\alpha A_m = \frac{\partial A_m}{\partial X^\alpha} - \mathcal{D}_m \Omega \ , \tag{2.26}$$

where α is an index running over all generalized coordinates X^α, then Eq. (2.24) implies that $\mathcal{D}^2 \Omega = \mathcal{D}^m \frac{\partial A_m}{\partial X^\alpha}$.

In order to count the number of solutions to Eq. (2.25) one then uses the Atiyah-Singer index theorem and obtains [8] that for $SU(N)$ instantons

$$\dim_{\mathbb{R}} \mathfrak{M}_k = 4kN \ , \tag{2.27}$$

in accordance with our observations so far.

2.2.2 Moduli space metric and quaternionic structure

The next step is to introduce a metric on the moduli space component $\mathfrak{M}_k$ implicitly used above when defining the orthogonality condition. It turns out that

$$g_{\alpha\beta} = \langle \delta_\alpha A, \delta_\beta A \rangle = -2 \int \mathrm{d}^4 x \, \mathrm{Tr} \left(\delta_\alpha A^m \delta_\beta A_m \right) \tag{2.28}$$

is a proper metric containing information crucial to evaluating the path integral. Recalling that for our toy model we had to integrate over all values of the generalized coordinate t_1 (see Eq. (1.11)), it is natural to expect an integral over the instanton moduli space. Since the metric (2.28) provides $\mathfrak{M}_k$ with a natural volume-form, it is the quantity we need when integrating over all zero modes.

In addition to being a Riemannian manifold, $\mathfrak{M}_k$ turns out to have even more structure. In fact, it is an, albeit singular, complex Hyper-Kähler manifold, meaning that there are three different complex structures $\mathbf{I}^a$, $a = 1, 2, 3$ satisfying

$$\mathbf{I}^a \mathbf{I}^b = -\delta^{ab} + \epsilon^{abc} \mathbf{I}^c \ . \tag{2.29}$$

The key to obtaining the different complex structures is noting that Eq. (2.25) has two free indices $\dot\alpha$ and $\dot\beta$. Given a zero mode $\delta A_{\alpha\dot\alpha}$, $\delta A_{\alpha\dot\alpha} G^{\dot\alpha}_{\dot\beta}$ is also a zero mode for any 2×2 matrix G. Let us therefore define

$$\begin{aligned} \left(\mathbf{I}^a \cdot \delta_\mu A \right)_{\alpha\dot\alpha} &= \delta_\mu A_{\alpha\dot\beta} \left(i \sigma^{a\dot\beta}{}_{\dot\alpha} \right) \\ &= \delta_\nu A_{\alpha\dot\alpha} (\mathbf{I}^a)^\nu{}_\mu \ , \end{aligned} \tag{2.30}$$

where we view the second line of the above equation as a definition of the coefficients

$(\mathbf{I}^a)^{\nu}{}_{\mu}$. Their existence is guaranteed, since the zero modes are solution to the linear differential equation (2.25) and must therefore form a vector space. With this definition, the three $4kN \times 4kN$ matrices $(\mathbf{I}^a)^{\nu}{}_{\mu}$ satisfy Eq. (2.29).
In Section 2.4 we will then see that these structures are integrable as well.

2.3 A stringy motivation of the ADHM construction

After this detour on the geometry of the instanton moduli space, it is time to return to the question raised in Section 2.1.3: How can one find and classify solutions to Eq. (2.13) for arbitrary values of k and any gauge group $SU(N)$? At first sight, solving the system of non-linear PDEs may look daunting, but, surprisingly, only three years after the discovery of the BPST instanton, Atiyah, Drinfeld, Hitchin and Manin found a way to construct all possible instanton solutions by reducing Eq. (2.13) to a system of algebraic equations using twistor spaces and complex algebraic geometry [3].
In this section we review their construction, although we do not present it in their original form. Rather, we first follow Douglas' interpretation of the ADHM construction in terms of brane configurations in Type II string theory [13, 14]. Background information on how to construct different classes of supersymmetric gauge theories on D-branes can e.g. be found in [18].

2.3.1 A system of branes

The key to seeing how instantons arise from stringy brane constructions lies in thinking of instantons as codimension four objects. An instanton solution embedded in $p+1$ dimensions will then be a solitonic object extending in $p-3$ spacetime directions. A natural setting for such a scenario is given by a set of Dp- and $D(p-4)$-branes in Type II string theory and, as the following heuristic argument suggests, it gives rise to instantons as we know from our familiar gauge theory on $\mathbb{R}^4$.
The low energy effective theory living on the world-volume of the Dp-branes is $U(N)$ SYM-theory with 16 supercharges in $p+1$ dimensions with certain couplings between fields localized on the branes and those living in the bulk of spacetime. Douglas showed

	0	1	2	3	4	5	6	7	8	9
D3	x	-	-	-	-	x	x	x	-	-
D7	x	x	x	x	x	x	x	x	-	-

Table 1: Configuration of the D3-D7 brane system. Crosses indicate that the object is extended in the respective direction.

Field	$U(k)_{\text{gauge}}$	$U(N)_{\text{global}}$	$SU(2)_R \times U(1)_R$
Φ	$[1,0,\ldots,0,1]_0$	$[0,\ldots,0]_0$	$[0]_0$
(B_1, B_2)	$[1,0,\ldots,0,1]_0$	$[0,\ldots,0]_0$	$[1]_1$
I	$[1,0,\ldots,0]_1$	$[0,\ldots,1]_{-1}$	$[0]_1$
J	$[0,\ldots,0,1]_{-1}$	$[1,0,\ldots,0]_1$	$[0]_1$

Table 2: Scalar field content of the Dp-$D(p-4)$ brane system with representations denoted by their Dynkin labels.

that there is a term

$$\int \mathrm{d}^{p+1}x \, C^{(p-3)} \wedge F \wedge F \, . \tag{2.31}$$

Recalling that instantons are objects with non-zero localized instanton charge density $F \wedge F \approx c \cdot \delta^{(4)}(x)$ for some constant c , an instanton in the gauge theory on the Dp-branes will roughly contribute a term proportional to

$$\int \mathrm{d}^{p-3}x \, C^{(p-3)} \tag{2.32}$$

to the total action, where the integral ranges over some $p-3$-dimensional subspace. This, however, is the same contribution that a D$(p-4)$-brane sourcing the Ramond-Ramond field $C^{(p-3)}$ would give! Working out the precise factors, this can rigorously be shown to be true.

In principle, one could now consider various choices of p to examine the implications of this new point of view, but to study the vacuum structure of our theory, let us for concreteness take $p = 7$, since it gives the sort of $3+1$-dimensional field theory on the $D(p-4)$-branes that we are the most comfortable with. We choose a brane configuration with k D3-branes and N D7-branes and position them as listed in Table 1. To examine the gauge theory living on the set of D3-branes, one should first determine the field content. Without the presence of the D7-branes, the theory would be an $\mathcal{N} = 4$ $U(N)$

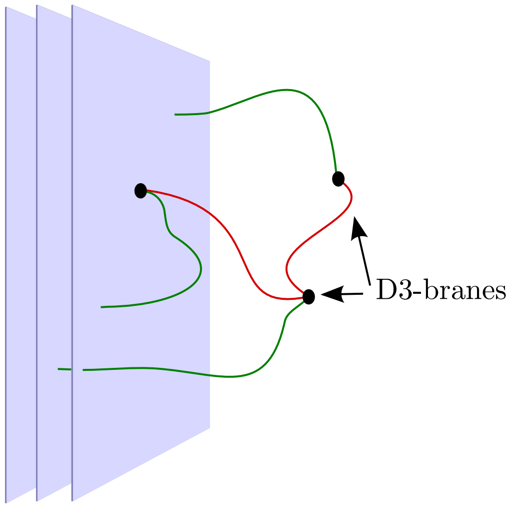

Figure 3: Brane set-up with (B_1, B_2) in visualized as red strings and (I, J) sketched as green strings. D3-branes are shown as black dots.

SYM theory with one $\mathcal{N} = 4$ vector multiplet. In $\mathcal{N} = 1$ language, this amounts to one vector multiplet V and three chiral multiplets B_1, B_2 and $\varPhi$ in the adjoint representation. The D7-branes, on the other hand, break another half of the supersymmetry by giving rise to another set of matter fields, namely two chiral $\mathcal{N} = 1$ multiplets, transforming in the (anti-)fundamental representation of $U(k)$. These can be visualized as strings stretching from a D7-brane to a D3-brane and hence carry two Chan-Paton factors ranging from 1 to k and 1 to N respectively. The additional $U(N)$ symmetry introduced by the D7-branes acts globally due to the fact that the D7-branes are extended in more directions than the D3-branes and fields localized on them appear static from the point of view of an observer on a D3-brane.

Summing up, we find an $\mathcal{N} = 2$ supersymmetric gauge theory with the scalar matter content and R-symmetry given by Table 2. Since the set-up does not introduce further masses, we can write down the Lagrangian density up to a coupling constant g:

$$\begin{aligned}\mathcal{L} = &\int \mathrm{d}^4\theta \ \mathrm{Tr}\left(\varPhi^\dagger \mathrm{e}^{-2V}\varPhi + B_1^\dagger \mathrm{e}^{2V} B_1 + B_2^\dagger \mathrm{e}^{-2V} B_2\right) \\ &+ \int \mathrm{d}^2\theta \left(\mathrm{Tr}(J\varPhi I) + \mathrm{Tr}(B_1[\varPhi, B_2]) - \frac{1}{2g^2}\mathrm{Tr}(W_\alpha W^\alpha)\right) + \mathrm{h.c.}\end{aligned} \tag{2.33}$$

For our purposes, the gauge coupling can be ignored and after some manipulations one obtains the following scalar potential:

$$V = \mathrm{Tr}\left(|[\Phi, B_1]|^2 + |[\Phi, B_2]|^2 + |\Phi I|^2 + |J\Phi|^2\right) + \\ \mathrm{Tr}\left(|[\Phi, \Phi^\dagger] + [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J|^2 + |[B_1, B_2] + IJ|^2\right) \tag{2.34}$$

Before trying to find all possible vacua, it makes sense to look for the physical meaning of the various scalar fields appearing in the potential. Since the complex scalar fields B_1, B_2 and Φ all descended from the $\mathcal{N} = 4$ vector multiplet, they can be understood as parametrizing the coordinates of the D4-branes in the space transverse to their world-volumes. Because of its transformation behavior under the R-symmetry, Φ is on a different footing than B_1 and B_2. Its real and imaginary parts must therefore be associated with the position of the D3-branes transverse to the D7-branes, i.e. $\Phi = X_8 + iX_9$, while $B_1 = X_1 + iX_2$ and $B_2 = X_3 - iX_4$ are coordinates on the world-volume of the D7-branes.

In order to set Eq. (2.34) to zero, there are two options:

1. Set $B_1 = B_2 = 0$, $I = J^\dagger = 0$ and choose Φ to be diagonal. This part of the moduli spaces is called the *Coulomb branch* of the moduli space and it corresponds to the D3-branes being transversely separated from the D7-branes with a $U(1)^k$ gauge theory on their world-volume.

2. Set $\Phi = 0$ and satisfy:

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0 \tag{2.35}$$

$$[B_1, B_2] + IJ = 0 \tag{2.36}$$

With the above choice of vacua, all gauge symmetry is broken and after dividing out the $U(k)$ gauge transformations one therefore calls this part of the moduli space the *Higgs branch*:

$$\mathcal{M}_{\mathrm{Higgs}} \cong \{I, J, B_1, B_2 | V(I, J, B_1, B_2, \Phi = 0) = 0\}/U(k) \tag{2.37}$$

The Higgs branch describes the situation in which the D3-branes lie on top of the

D7-branes, which is precisely when we would expect them to appear as instantons in the D7-word-volume theory.

Indeed, Equations (2.35) and (2.36) are precisely the equations appearing in the original ADHM construction and they prove that

$$\mathcal{M}_{\text{Higgs}} \cong \mathfrak{M}_{inst} \; . \tag{2.38}$$

2.3.2 Constructing instanton solutions

Even though the statement in Eq. (2.38) is already very powerful, since it provides much geometrical information about $\mathfrak{M}_{inst}$, Atiyah, Drinfeld, Hitchin, and Manin went considerably further in their analysis. We now show how to explicitly construct instanton solutions. In order to do so, we rewrite our fields in a form more akin to that usually used when dealing with the ADHM construction.

Let i and j be Latin indices running from 1 to k and let u run from 1 to N. Then, neglecting whether indices are downstairs or upstairs, let

$$w_{uj\dot{1}} = I^j{}_u \quad \text{and} \quad w_{uj\dot{2}} = J^{\dagger j}{}_u \quad \text{or} \quad w_{uj} = \begin{pmatrix} I^j{}_u \\ J^{\dagger j}{}_u \end{pmatrix} \; . \tag{2.39}$$

In terms of w, the real equation (2.35) and the complex equation (2.36) can be repackaged into the $k \times k$ matrix equations

$$\sum_{u=1}^{N} w_u^\dagger \sigma^i w_u - i[X_m, X_n]\bar{\eta}^a_{mn} = 0 \quad a = 1,2,3 \; , \tag{2.40}$$

where $\bar{\eta}^a_{mn}$ are the anti-self-dual ’t Hooft symbols[1]. Now define the $(N+2k)\times 2k$ matrix Δ as

$$\Delta = \begin{pmatrix} w^T \\ X_m\sigma^m + x_m\sigma^m \end{pmatrix} \; . \tag{2.41}$$

Δ has the nice property that the product with its Hermitean conjugate factorizes in the

[1]The self-dual and anti-self-dual ’t Hooft symbols are defined as $\eta^a_{mn} = \epsilon^a{}_{mn4} + \delta^a_m\delta_{n4} - \delta^a_n\delta_{m4}$ and $\bar{\eta}^a_{mn} = \epsilon^a{}_{mn4} - \delta^a_m\delta_{n4} + \delta^a_n\delta_{m4}$, respectively.

spinor indices, i.e.

$$\begin{aligned}\Delta^\dagger \Delta &= w^\dagger w + X^\dagger X + X^\dagger x + x^\dagger X + x^\dagger x \\ &= f^{-1} \otimes \mathbb{1}_2 \end{aligned} \tag{2.42}$$

for some $k \times k$ matrix f. Factorization of the terms linear and quadratic in x can be seen using the identity $\bar{\sigma}^{(m}\sigma^{n)} = \delta^{mn}\mathbb{1}_2$. For the terms independent of x, requiring factorization is equivalent to asking that $\mathrm{Tr}\left(\sigma^a(w^\dagger w + X^\dagger X)\right)$ vanishes for $a = 1, 2, 3$, which is guaranteed by comparing

$$\mathrm{Tr}\left(\sigma^a(ww^\dagger + X^\dagger X)\right) = \tag{2.43}$$

$$\mathrm{Tr}\left(w^\dagger \sigma^a w + X_m^\dagger X_n \bar{\sigma}^m \sigma^a \sigma^n\right) \tag{2.44}$$

to Eq. (2.40) and using that $\bar{\sigma}_m \sigma^a \sigma_n = 2i\bar{\eta}^a_{mn}$.

Even though it does not look like we have gained much, we are almost there. Considering Δ as a map from $\mathbb{C}^{N+2k}$ to $\mathbb{C}^{2k}$, its kernel $\ker(\Delta)$ is an N-dimensional complex vector space. Now pick U be an $(N+2k) \times N$ matrix that contains a suitably normalized basis of $\ker(\Delta)$, i.e.

$$\Delta^\dagger U = 0 \quad \text{and} \quad U^\dagger U = \mathbb{1}_N \ , \tag{2.45}$$

then the gauge field

$$A_m = U^\dagger \partial_m U \tag{2.46}$$

is a $SU(N)$ instanton of charge k.

To see that Eq. (2.46) indeed defines an instanton solution one calculates

$$\begin{aligned} F_{mn} &= \partial_m A_n - \partial_n A_m + [A_m, A_n] = \partial_{[m} A_{n]} + A_{[m} A_{n]} \\ &= \partial_{[m} U^\dagger \partial_{n]} U + U^\dagger (\partial_{[m} U) U^\dagger (\partial_{n]} U) \\ &= \partial_{[m} U^\dagger \partial_{n]} U - (\partial_{[m} U^\dagger) U U^\dagger (\partial_{n]} U) \\ &= \partial_{[m} U^\dagger (1 - U U^\dagger) \partial_{n]} U \end{aligned} \tag{2.47}$$

and uses that by combining Equations (2.42) and (2.46) $UU^\dagger$ can be shown to be a projection operator satisfying

$$UU^\dagger \Delta = U(\Delta^\dagger U)^\dagger = 0 \quad \text{and} \quad UU^\dagger U = U \ . \tag{2.48}$$

The same holds for $1 - \Delta f \Delta^\dagger$ as

$$(1 - \Delta f \Delta^\dagger)\Delta = \Delta - \Delta f f^{-1} = 0 \quad \text{and} \quad (1 - \Delta f \Delta^\dagger) U = U \tag{2.49}$$

and therefore $UU^\dagger = 1 - \Delta f \Delta^\dagger$. One can therefore simplify (2.47) further by writing

$$\begin{aligned} F_{mn} &= \partial_{[m} U^\dagger \Delta f \Delta^\dagger \partial_{n]} U \\ &= (\partial_{[m} U^\dagger) \Delta f \Delta^\dagger \partial_{n]} U + U^\dagger \partial_{[m} \Delta f \Delta^\dagger \partial_{n]} U \\ &= U^\dagger \partial_{[m} \Delta f \partial_{n]} (\Delta^\dagger) U + U^\dagger \partial_{[m} \Delta f \Delta^\dagger \partial_{n]} U \\ &= U^\dagger \partial_{[m} \Delta f \partial_{n]} \Delta^\dagger U \\ &= U^\dagger \sigma_{[m} f \bar{\sigma}_{n]} U \\ &= U^\dagger f \sigma_{[m} \bar{\sigma}_{n]} U \\ &= 4 U^\dagger f \sigma_{mn} U \ . \end{aligned} \tag{2.50}$$

Since σ_{mn} is clearly self-dual, so is the field strength F_{mn}.

Having shown that one obtains self-dual field strengths by the above method, it remains to be seen that this procedure gives the right instanton charge. Doing so requires more matrix algebra and, in particular, an identity discovered by Osborn [33]. Additionally, one can show that the above construction provides all instanton solution. For sake of brevity, we will do neither here and refer e.g. to [12] for more details.

2.4 The instanton moduli space revisited

In the previous section we introduced the instanton moduli space as a certain branch of the moduli space of a supersymmetric gauge theory. Let us now explore this new point of few a bit further and see how reconcile the notions of a metric and the complex structures introduced in Section 2.2 with the ADHM construction.

2.4.1 The instanton moduli space as a Hyper-Kähler quotient

Defining the instanton moduli space as in Eq. (2.37) by restricting and then quotienting a bigger space, is called a Hyper-Kähler quotient. This definition has certain advantages over considering zero modes of some operator as it gives the possibility to derive geometrical properties from a parent space that might have simpler structure. To understand this procedure better, let us quickly review some properties of Hyper-Kähler spaces.

Any complex manifold $\mathcal{M}$ admits a complex structure $\mathbf{I}$, which is a linear map acting on the tangent space $T_x\mathcal{M}$ that squares to -1, i.e. $\mathbf{I}^2 = -\mathbb{1}$ and is integrable:

$$[\mathbf{I}X, \mathbf{I}Y] - [X, Y] - \mathbf{I}[X, \mathbf{I}Y] - \mathbf{I}[\mathbf{I}X, Y] = 0 \quad \forall\, X, Y \in T_x\mathcal{M} \tag{2.51}$$

A metric g on $\mathcal{M}$ is called Hermitian if it satisfies $g(\mathbf{I}X, \mathbf{I}Y) = g(X, Y)$ and it defines a fundamental 2-form ω

$$\omega(X, Y) = g(\mathbf{I}, Y)\ . \tag{2.52}$$

Its definition implies that ω is antisymmetric and, picking complex coordinates on $\mathcal{M}$, I, g and ω can locally be written as

$$\mathbf{I} = \begin{pmatrix} i\,\delta^i{}_j & 0 \\ 0 & -i\,\delta^{\bar{i}}{}_{\bar{j}} \end{pmatrix}, \quad g = g_{i\bar{j}}\mathrm{d}z^i\mathrm{d}\bar{z}^{\bar{j}} \quad \text{and} \quad \omega = ig_{i\bar{j}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}}\ . \tag{2.53}$$

If $d\omega = 0$, then $\mathcal{M}$ is called Kähler, ω its Kähler form and the metric can be derived from a Kähler potential K via

$$g_{i\bar{j}} = \frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^{\bar{j}}} K \ . \tag{2.54}$$

Furthermore, it can be shown that a Kähler manifold $\mathcal{M}$ of real dimension $4n$ has a reduced holonomy group of $U(2n)$. As mentioned before, a Hyper-Kähler manifold has even more structure, namely it has three complex structures obeying the quaternionic algebra of Eq. (2.29). If $\mathcal{M}$ is Hyper-Kähler, then the holonomy is reduced even further and the tangent space $T_x\mathcal{M}$ admits a $SU(2) \times Sp(n)$ structure. In terms of real coordinates x^μ $\mu = 1 \dots 4n$ on $\mathcal{M}$ the metric can be written as

$$g = h^{i\dot{\alpha}}{}_\mu h^{j\dot{\beta}}{}_\nu \, \Omega_{ij} \, \epsilon_{\dot{\alpha}\dot{\beta}} \, \mathrm{d}x^\mu \mathrm{d}x^\nu \tag{2.55}$$

and the three complex structures with their respective Kähler forms as

$$(\mathbf{I}^{(c)} \cdot h)^{i\dot{\alpha}}{}_\mu = -i\sigma^{(c)\dot{\alpha}}{}_{\dot{\beta}} h^{i\dot{\beta}}{}_\mu \quad \text{and} \quad \omega^{(c)} = i\sigma^{(c)\dot{\alpha}}{}_{\dot{\gamma}} h^{i\dot{\beta}}{}_\mu h^{j\dot{\gamma}}{}_\nu \, \Omega_{ij} \, \epsilon_{\dot{\alpha}\dot{\beta}} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \ . \tag{2.56}$$

In the above expressions Ω_{ij} is an $Sp(n)$ 2-form and i, j run from 1 to $2n$.
Let us now see how the Hyper-Kähler quotient works: Its basic idea is the one already introduced in Eq. (2.37). We are interested in Hyper-Kähler spaces $\mathcal{M}$ admitting isometries that form some lie group G. Pick a set of Hermitian generators T^r $\quad r = 1 \dots \dim G$, whose action on the tangent space defines vector fields X_r in the usual way. If G preserves both metric and complex structure, then

$$\mathcal{L}_{X_r} g = \mathcal{L}_{X_r} \mathbf{I}^{(c)} = 0 \quad \text{for} \quad c = 1, 2, 3 \, , \tag{2.57}$$

which implies that $\mathcal{L}_{X_r}\omega^{(c)} = 0$, since ω is defined solely in terms of $\mathbf{I}^{(c)}$ and g. But since $\mathcal{L}_X = \iota_X \mathrm{d} + \mathrm{d}\iota_X$ for Lie derivatives of forms, $\mathcal{L}_{X_r}\omega^{(c)} = \mathrm{d}(\iota_{X_r}\omega^{(c)}) = 0$ and there exist *Hamiltonian* functions $\mu_{(c)}$ such that

$$\iota_{X_r}\omega^{(c)} = \mathrm{d}\mu^{X_r}_{(c)} \ . \tag{2.58}$$

If one requires further that these transform *equivariantly*, i.e. $X[\mu^Y_{(c)}] = \mu^{[X,Y]}_{(c)} \quad \forall X, Y,$

then $\mu^{X_r}_{(c)}$ are called *moment maps*. One can arrange the moment maps to take values in the Lie algebra $\mathfrak{g}$ of G to obtain

$$\mu_{(c)} = \sum_{r=1}^{\dim G} \mu^{X_r}_{(c)} T^r \ . \tag{2.59}$$

The actual quotienting consists of two steps. First, one restricts to the *level set* $\{x \in \mathcal{M} | \mu_{(c)}(x) = 0 \,\forall c\}$. After that, one further quotients this set by the group G to obtain

$$\mathcal{N} = \vec{\mu}^{-1}(0)/G \ , \tag{2.60}$$

where we grouped the three maps $\mu_{(c)}$ into a vector $\vec{\mu}$. $\mathcal{N}$ is then another Hyper-Kähler space of dimension $\dim M - 4 \dim G$.

By direct comparison to our treatment of the instanton moduli space in terms of a supersymmetric gauge theory, we see that this is exactly the same procedure that we encountered above. While the gauge group G corresponds to isometries of $\mathcal{M}$, the three moment maps are just the conditions for minimizing the scalar potential. Eq. (2.35) is one moment map, while (2.36), being a complex equation, gives rise to two real moment maps.

What is left to see is how to take advantage of this construction in order to find the metric $\mathfrak{M}_{inst}$ inherits from its mother space. Let us perform a small change notation and denote by $\tilde{\mathcal{M}}$ the mother space, by $\mathcal{M}$ the level set and by $\mathcal{N}$ the final quotiented space. Then it turns out that locally one can divide $T\tilde{\mathcal{M}}$ into

$$T\tilde{\mathcal{M}} = \mathcal{P} \oplus \mathcal{Q} \ , \tag{2.61}$$

where $\mathcal{Q}$ is the $3 \cdot \dim G$ dimensional linear hull of $\mathbf{I}^{(c)} X_r$ and $\mathcal{P}$ is its orthogonal complement. Viewing $\mathcal{M}$ as a G-bundle over $\mathcal{M}/G \cong \mathcal{N}$, $\mathcal{P}$, in turn, can locally be interpreted as the tangent space $T\mathcal{M}$. $T\mathcal{M}$ can again be divided into two parts, namely the $\dim G$-dimensional space $\mathcal{V}$ spanned by the vectors X_r and its orthogonal complement

$\mathcal{H}$. Finally, one therefore has

$$T\tilde{\mathcal{M}} = \mathcal{H} \oplus \mathcal{V} \oplus \mathcal{Q} \,. \tag{2.62}$$

The tangent space of $\mathcal{N}$, our instanton moduli space, is then $T\mathcal{M}/\mathcal{V} \cong \mathcal{H}$. The crucial point here is that elements of $T\mathcal{N}$ have a unique lift to $\mathcal{H} \subset T\tilde{\mathcal{M}}$ and therefore the metric on $\mathfrak{M}_{inst}$ is simply the pull-back of the flat metric on $\mathbb{R}^{4(kN+k^2)}$ which reads schematically

$$ds^2 = \mathrm{Tr}\left(|dI|^2 + |dJ|^2 + |dB_1|^2 + |dB_2|^2\right) \,. \tag{2.63}$$

Alternatively, for the special class of Hyper-Kähler spaces whose three Kähler forms share the same potential, one can introduce a Hyper-Kähler potential χ. Since flat Euclidean space whose dimension is a multiple of 4 naturally carries such a structure, it has a trivial Hyper-Kähler potential

$$\chi \sim \mathrm{Tr}\left(|I|^2 + |J|^2 + |B_1|^2 + |B_2|^2\right) \,. \tag{2.64}$$

By simply inserting parametrization of the coordinates I, J, B_1 and B_2 in terms of coordinates on $\mathfrak{M}_{inst}$ one then obtains the Hyper-Kähler potential of the instanton moduli space. Doing all the ADHM algebra rigorously, χ can be shown to be

$$\chi = -\frac{1}{4}\int \mathrm{d}^4x\, x^2\, \mathrm{Tr}(F_{mn}^2) \,, \tag{2.65}$$

which is also known as Maciocia's potential [27]. Following Maciocia's analysis, one can show that the metric and the complex structures obtained from χ are equal to those introduced in the previous section.

2.4.2 Singularities and their resolution

As we saw very briefly by looking at the BPST instanton in Eq. (2.20), the solution stops being well behaved as the size ρ of the instanton goes to zero. In order to see what effect this has on the instanton moduli space, let us take this concrete example to calculate the metric on $\mathfrak{M}_1$ for the $SU(2)$ instanton. To do so, one simply has to take derivatives of Eq. (2.20) with respect to the collective coordinates and check that the gauge fixing condition Eq. (2.24) is met. In this case, there are $4+1+3=8$ collective coordinates:

1. For the positions X^m the respective zero mode is just the field strength, i.e.

$$\delta_m A_n = F_{mn} \ , \tag{2.66}$$

since its gauge fixing condition are just the equations of motion for the field strength. One finds that

$$-\int \mathrm{d}^4x \ \mathrm{Tr}\,(\delta_m A_r \delta_n A^s) = 4\, S_{\text{inst}}\, \delta_{mn} \ . \tag{2.67}$$

2. The zero mode corresponding to the size of the instanton, $\delta A_m = \frac{\partial A_m}{\partial \rho}$ already fulfills the gauge fixing condition and

$$-\int \mathrm{d}^4x \ \mathrm{Tr}\,(\delta A_m \delta A^m) = 8\, S_{\text{inst}} \ . \tag{2.68}$$

3. Lastly, one can choose the zero modes for the global gauge transformations to be $\delta_i A_m = \mathcal{D}_m \Lambda_i$ where Λ_i must not vanish at infinity. In particular, $\Lambda_i = \frac{2(x-X)^2}{((x-X)^2+\rho^2)^2}\sigma^i$ satisfy (2.24) and yield

$$-\int \mathrm{d}^4x \ \mathrm{Tr}\,(\delta_i A_m \delta_j A^m) = 8\, S_{\text{inst}}\, \rho^2 \ . \tag{2.69}$$

All other components of the metric vanish, but there is one more caveat: Since A_m transforms according to Eq. (2.3) under gauge transformations, it is invariant under the $\mathbb{Z}_2 \subset SU(2)$ and therefore the inequivalent global gauge transformations form $SU(2)/\mathbb{Z}_2 \cong S_3/\mathbb{Z}_2$ instead of S_3. Pairing up S_3 with the $\mathbb{R}^+$ parametrized by ρ, one arrives at

$$\mathfrak{M}_{k=1,SU(2)} = \mathbb{R}^4 \times \mathbb{R}^4/\mathbb{Z}_2 \ . \tag{2.70}$$

Since $\mathbb{Z}_2$ does not act freely on the origin of $\mathbb{R}^4$, the instanton moduli space has a singularity associated to the instanton of zero size. It turns out that this is a general feature even also for $k>1$. Since instantons of charge k can at least in the limit of large separation be considered as k singly charged instantons, the singularities of $\mathfrak{M}_k$ become worse for increasing k and can be shown to correspond to loci on which $1 \leq m \leq k$ instantons shrink to zero size.

Let us now look at how these singularities can be cured. Since instanton moduli spaces found rich applications in Mathematics not long after their discovery in Physics, there has been considerable interest in resolving their singularities. Here, however, we will concentrate on one particular resolution that can be physically motivated, even though we will give this motivation only a posteriori. An easy way of resolving the singularities of $\mathfrak{M}_{\text{inst}}$ is by reconsidering its Hyper-Kähler quotient discussed in the previous section. If one considers more generally

$$\mathfrak{M}_k^{\vec{\zeta}} = \vec{\mu}^{-1}(\vec{\zeta})/G \tag{2.71}$$

with $\vec{\zeta} = (\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)})$ where $\zeta^{(c)}$ takes values only in the Lie algebra of the $U(1)$ factor of G, then for non-zero $\vec{\zeta}$ one obtains smooth resolutions of $\mathfrak{M}_{\text{inst}}$ instead.
Let us see how this works for the $k = 1$ case of $G = U(N)$. On the D3-brane, we find a $U(1)$ gauge theory with N charged hypermultiplets and a neutral hypermultiplet which does not appear in the potential and can, as usual, be considered to parametrize the position of the instanton. Since $k = 1$, the matrices I and J become N-dimensional vectors. Choosing $\vec{\zeta} = (0, 0, \zeta)$, the modified ADHM constraints (2.40) read

$$\sum_{a=1}^{N} \Big(|I_a|^2 - |J_a|^2 \Big) = \zeta, \qquad \sum_{a=1}^{N} I_a J_a = 0 \tag{2.72}$$

and the $U(1)$ by which we have to quotient acts as

$$I_a \to \mathrm{e}^{i\alpha} I_a \quad \text{and} \quad J_a \to \mathrm{e}^{-i\alpha} J_a \ . \tag{2.73}$$

To identify this space, it is useful [40] to first set $J_a = 0$. Then, writing $I_a = x_a + iy_a$ with real x_a, y_a, (2.73) is simply the definition of a sphere S^{2N-1} with radius $\sqrt{\zeta}$. Quotienting by the $U(1)$ action reduces the sphere to complex projective space $\mathbb{CP}^{N-1}$ with size ζ. If one allows non-zero J_a, then the second condition of Eq. (2.73) ensures that J must be orthogonal to I. One thus obtains the cotangent bundle of $\mathbb{CP}^{N-1}$, and, adding the position of the instanton, one finds

$$\mathfrak{M}^{\zeta}_{k=1,U(N)} \cong \mathbb{R}^4 \times T^*\mathbb{CP}^{N-1} \ . \tag{2.74}$$

In particular, since $S^2 \cong \mathbb{CP}^1$, $\mathfrak{M}^{\zeta}_{k=1,U(2)} \cong \mathbb{R}^4 \times T^*S^2$, which turns out to be the smooth resolution of the singular manifold $\mathbb{R}^4 \times \mathbb{R}^4/\mathbb{Z}_2$. In this special case, even the metric is known: It is the Eguchi-Hanson metric [15] and for $\zeta \to 0$ one recovers the original singularity.

After working through this concrete example and suggesting that the instanton moduli space singularities can be removed without much effort, let us show how to interpret the parameters $\vec{\zeta}$ physically. It turns out that they arise when considering instantons not on ordinary spacetime, but on a non-commutative space, namely one whose coordinates x^m satisfy

$$[x_m, x_n] = i\theta_{mn} \ . \tag{2.75}$$

An simple way of implementing such an algebra on a space is by replacing ordinary multiplication of function by a Moyal-type product defined as

$$f(x) \star g(x) = \exp\left(\frac{i}{2}\theta_{mn}\frac{\partial}{\partial x^m}\frac{\partial}{\partial y^n}\right) f(x)g(y)\bigg|_{x=y} \ . \tag{2.76}$$

Splitting θ_{mn} into

$$\theta_{mn} = \xi^i \eta^i_{mn} + \zeta^i \bar{\eta}^i_{mn} \ , \tag{2.77}$$

the instanton equation $F = \star F$ is affected only by the anti-self-dual part of θ_{mn}, while the converse statement holds for anti-instantons. Working through the whole ADHM construction and replacing ordinary multiplication by star-type multiplication, Nekrasov and Schwarz found [30] that the factorization requirement in Eq. (2.42) is modified to

$$\Delta^{\dagger} \star \Delta = f^{-1} \otimes \mathbb{1}_2, \tag{2.78}$$

which is satisfied exactly by the modified ADHM constraints considered above. There is much more that can be said on non-commutative geometry and interested readers can e.g. consult [31] for a review.

2.5 A word or two on Hilbert series

To conclude the chapter on instantons and their moduli spaces, we present another approach to take advantage of being able to regard the instanton moduli space as the moduli space of vacua of a certain supersymmetric gauge theory. Being described by algebraic equations, supersymmetric moduli spaces are predisposed to be treated by algebro-geometric methods [19]. Hence there may be much to be learned by studying the algebras of functions defined on them and for the past years there has been considerable activity in this area. More concretely, one can study the space of gauge-invariant BPS operators (GIO's) of a supersymmetric gauge theory and compute certain characteristic quantities. One of these quantities is called the *Hilbert series* and the remainder of this section will be dedicated to it.

Although to our knowledge there does not yet exist a comprehensive review on this topic, there are numerous original articles (for a small excerpt e.g. [19, 6, 16, 21]) introducing the relevant techniques while going much beyond this little peek at Hilbert series.

2.5.1 The Hilbert Series and Plethystics

One of the natural properties of a topological space is its dimension. In this case, the space of interest consists of multitrace GIO's, i.e. something like

$$\mathcal{O} = \mathrm{Tr}\left(A_{i_1} A_{i_2} \dots A_{i_{n_1}}\right) \cdot \ldots \cdot \mathrm{Tr}\left(A_{i_{n_{k-1}+1}} \dots A_{i_{n_k}}\right) \tag{2.79}$$

for some set of operators A_i. Unfortunately for us, this space, let us call it $\mathcal{M}$, is infinite-dimensional and there is no information to be gained. On the bright side, it does admit a natural grading by partitioning it into subspaces consisting of operators with a fixed number k of A_i's:

$$\mathcal{M} = \bigoplus_{k=1}^{\infty} \mathcal{M}_k \tag{2.80}$$

These subspaces turn out to be finite-dimensional and their dimensions $\dim \mathcal{M}_k$ constitute the first piece of information we are interested in. Furthermore, there is usually a set of global continuous symmetries of the theory and therefore the operators A_i must transform in some representation of the global symmetry group G. To refine our counting, we therefore aim to calculate the characters of the respective representations, i.e.

$$c_k \equiv \mathrm{Tr}_{\mathcal{M}_k}(h) \ , \tag{2.81}$$

where h is an arbitrary element of the global gauge group. The set $\{c_k\}$, which is generically infinite, can be packaged into a generating function

$$g(\{z_i(h)\};t) = \sum_{k=0}^{\infty} c_k t^k \ , \tag{2.82}$$

where t_k is a fugacity counting the number of operators, and z_i are weights taking values in the maximal torus of the global gauge group. Note that by setting h equal to the identity element of G, or equivalently $z_i = 1 \ \forall\, i = 1, \ldots, \mathrm{rk}\, G$, one recovers the dimension of $\mathcal{M}_k$. This function $g(\{z_i\};t)$ is called the Hilbert series.

In order to apply this to the ADHM construction, we must introduce two more tools. First of all, one needs a procedure to obtain the Hilbert series of multitrace operators given the Hilbert series for singletrace operators. Since the kth power of a singletrace operator in the representation R of G transforms under the symmetric product $\mathrm{Sym}^k[R]$, one has to look for a function generating symmetrizations. The function accomplishing that is called the *Plethystic exponential* and it is defined as

$$\mathrm{PE}[g(t_1, \ldots, t_k)] = \exp\left(\sum_{r=1}^{\infty} \frac{g(t_1^r, \ldots, t_k^r)}{r} \right) \ , \tag{2.83}$$

where $g(t_1, \ldots, t_k)$ is a multivariable function vanishing at the origin $(0, \ldots, 0)$.

The second part deals with realizing the constraints appearing when taking the Hyper-Kähler potential. Rewriting the problem in terms of the language of $\mathcal{N} = 1$ supersymmetric gauge theories, those constraints can be divided into two classes, namely F-Terms and D-terms. F-terms can be derived using a superpotential W while the D-terms originate from the generators of the gauge group. Schematically,

$$V = \sum_i |f_i|^2 + \frac{1}{2} g^2 \sum_a (D^a)^2 \tag{2.84}$$

where

$$f_i = \frac{\partial W}{\partial \phi_i} \quad \text{and} \quad D^a = \sum_i \phi_i^\dagger T^a \phi_i \tag{2.85}$$

and T^a are the (Hermitean) generators of the gauge group H. The vacuum moduli space consists of those field configurations satisfying $f_i = 0$ and $D^a = 0$ for all values of a and i. It can further be shown that the D-terms are simply gauge-fixing conditions and one can obtain the vacuum moduli space by quotienting the space of field configurations satisfying the F-term constraints only, $\mathcal{F}^\flat$ by the complexified gauge group $H_{\mathbb{C}}$, i.e.

$$\mathcal{M} = \mathcal{F}^\flat // H_{\mathbb{C}} \, . \tag{2.86}$$

Calculating the Hilbert series of the space of functions defined on the moduli space $\mathcal{M}$ (which, for simplicity we call the Hilbert series of $\mathcal{M}$) can therefore be divided into two smaller tasks. First, one can determine $g^{\mathcal{F}^\flat}$, the Hilbert series of the space of functions on $\mathcal{F}^\flat$. In a second step, one can then eliminate those functions not invariant under local gauge transformations to end up with only the GIO's that we are interested in.
In order to perform the first step, it is helpful if $\mathcal{F}^\flat$ is a complete intersection, i.e. the number of constraints equals the codimension of $\mathcal{F}^\flat$ when embedded in the space of all possible field configurations. If this is the case, then the Hilbert series can be expressed as a quotient of the Hilbert series generated by the embedding space of $\mathcal{F}^\flat$ by the Hilbert series generated by the constraints. We will see this in more detail below.

To eliminate gauge-dependent operators, recall from Group theory that there is a natural inner product on the space of character functions and that the characters of irreducible representations are orthogonal with respect to this inner product, i.e.

$$\begin{aligned} \langle \chi_{R_i}(\{z_i\}), \chi_{R_j}(\{z_j\})\rangle_H &\equiv \int \mathrm{d}\mu_H \, \chi_{R_i}(\{z_i\})^* \chi_{R_j}(\{z_j\}) \\ &= \delta_{ij} \ , \end{aligned} \tag{2.87}$$

where $\mathrm{d}\mu_H$ is the Haar measure of the group H and $\{z_i\}$ are the fugacities taking values in $U(1)^{\mathrm{rk}\, H}$. Since the character of the trivial representation is by definition 1, the projection onto the subspace of gauge-invariant operators simply reads

$$g^{\mathcal{M}}(\{y_i\}; t) = \int \mathrm{d}\mu_H \, g^{\mathcal{F}^\flat}(\{y_i\}; \{z_i\}; t) \ , \tag{2.88}$$

where $\{y_i\}$ and $\{z_i\}$ are the respective fugacities of the global symmetry group G and the local gauge group H and the integration is with respect to the latter group.

2.5.2 Reproducing the results for $SU(N)$

Having dealt with the formalism, let us try and apply it to the case of $SU(N)$ instantons. As explained in Section 2.3, the relevant gauge theory is an $\mathcal{N} = 2$ gauge theory with the scalar field content of Table 2. Let us assign fugacities $\{y_i\}$ to the global symmetry group $U(N)$, fugacities $\{z_i\}$ to the local gauge symmetry $U(k)$, fugacity x to the $SU(2)$ R-symmetry and t to $U(1)_R$. Note that t will also be counting the number of fields since Φ will be set to zero on the Higgs branch relevant to our computations. Furthermore, one observes that the $U(1)$ charges of the fields with respect to the $U(1) \subset U(k)$ and the $U(1) \subset U(N)$ are related by a minus sign. We can therefore drop the latter since it contains no additional information.

From Equation (2.33), one easily reads off the superpotential

$$W = \mathrm{Tr}\,(J\Phi I) + \mathrm{Tr}\,(B_1[\Phi, B_2]) \tag{2.89}$$

and, since $\Phi = 0$ everywhere on the Higgs branch, the only relevant equation is

$$0 = \frac{\partial W}{\partial \Phi^i{}_j} = \sum_{u=1}^{N} J^u{}_i I^j{}_u + [B_1, B_2]^j{}_i \ . \tag{2.90}$$

Either from Equation (2.27) or from counting degrees of freedom on the Higgs branch of this theory we have that the real dimension of the $SU(N)$ instanton moduli space is $4kN$. The hypermultiplets transforming in the bi-fundamental representation of $U(N) \times U(k)$ have $2kN$ complex degrees of freedom and the hypermultiplets from the $\mathcal{N} = 4$ vector multiplet give another $2k^2$ complex degrees freedom. On the other hand, both the F-terms and the D-terms contain k^2 complex equations and therefore $\mathcal{M}^{\text{Higgs}} \cong \mathfrak{M}_{\text{inst}}$ is indeed a complete intersection.

We can now write down the Hilbert series of $\mathcal{F}^\flat$:

$$g^{\mathcal{F}^\flat}_{k,SU(N)}(x; \{y_i\}; \{z_a\}; t) = \frac{\text{PE}[\sum_{a=1}^k z_a [0, \dots, 0, 1]_y t + \sum_{a=1}^k z_a^{-1} [1, 0, \dots, 0]_y t]}{\text{PE}[\sum_{a,b=1}^k z_a \cdot z_b^{-1} \, t^2]} \times \text{PE}[\sum_{a,b=1}^{k} z_a \cdot z_b^{-1} [1]_x t] \tag{2.91}$$

In writing down this equation we used that the character of the fundamental representation of $U(k)$ can be written as $\sum_{j=1}^k z_k$ and that $\text{PE}[f(t) + g(t)] = \text{PE}[f(t)] \cdot \text{PE}[g(t)]$. The denominator is determined by Eq. (2.90), which transforms in the adjoint representation of $U(k)$ and is second order in the fields. Note that we denote characters χ_R of representations R simply by the Dynkin labels of the representation.

Using that $[1, 0, \dots, 0]_y = \sum_{i=1}^N \frac{y_i}{y_{i-1}}$ for $y_0 = y_N = 1$ and hence $[1, 0, \dots, 0]_y = \sum_{i=1}^N \frac{y_{i-1}}{y_i}$, we can evaluate the Plethystic exponential. Since $\text{PE}[t] = \frac{1}{1-t}$, one obtains

$$g^{\mathcal{F}^\flat}_{k,SU(N)} = \frac{\prod_{1 \le a,b \le k} \left(1 - \frac{z_a}{z_b} t^2\right)}{\left[\prod_{i=1}^N \prod_{a=1}^k \left(1 - t z_a \frac{y_{i-1}}{y_i}\right)\left(1 - \frac{t}{z_a} \frac{y_i}{y_{i-1}}\right)\right] \left[\prod_{\delta=\pm1} \prod_{1 \le a,b \le k} \left(1 - \frac{z_a}{z_b} t x^\delta\right)\right]} \ . \tag{2.92}$$

The last step is to integrate over the local group $U(k)$

$$g^{\text{Higgs}}_{k,SU(N)}(x; \{y_i\}; t) = \int \mathrm{d}\mu_{U(k)} \ g^{\mathcal{F}^\flat}_{k,SU(N)} \ , \tag{2.93}$$

where the Haar measure for $U(k)$ is given by

$$\int \mathrm{d}\mu_{U(k)} = \frac{1}{k!}\frac{1}{(2\pi i)^k}\oint_{|z_1|=1}\frac{1}{z_1}\dots\oint_{|z_k|=1}\frac{1}{z_k}\prod_{a,b=1}^{k}(1-z_a)\left(1-z_b^{-1}\right) \ . \tag{2.94}$$

Equation (2.93) with the expression for g^{Higgs} inserted is sometimes called the *Molien-Weyl* formula.

In principle one could now evaluate the multiple contour integrals using the residue theorem. Practically, though, this becomes very challenging even for small values of k. The cases $k = 1$ and $k = 2$ were treated in [7] and [20], but we quote only the former result. Evaluating the integral above, one finds for a $SU(N)$ instanton of charge $k = 1$ that its Hilbert series can be written as

$$g^{\text{Higgs}}_{k=1,SU(N)}(x;\{y_i\};t) = \frac{1}{(1-xt)\left(1-\frac{t}{x}\right)}\sum_{k=0}^{\infty}[k,0,\dots,0,k]_{SU(N)}\,t^{2k} \ . \tag{2.95}$$

One observes that the Hilbert series of the instanton moduli space factorizes and, indeed, this is the expected behavior. Since we noted above that

$$\begin{aligned}\mathfrak{M}_{\text{inst}} &\cong \mathbb{R}^4 \times \tilde{\mathfrak{M}}_{\text{inst}} \\ &\cong \mathbb{C}^2 \times \tilde{\mathfrak{M}}_{\text{inst}} \ ,\end{aligned} \tag{2.96}$$

where the first factor describes the position of the instanton in spacetime. $\mathbb{C}^2$ has a symmetry group $U(2)$ and this symmetry group gives a factor of $\frac{1}{(1-xt)\left(1-\frac{t}{x}\right)}$.
Lastly, one can check that the Hilbert series gives the right dimension of the moduli space. Unrefining the Hilbert series by setting $x = y_i = 1$, one obtains [7]

$$g^{\text{Higgs}}_{k=1,SU(N)}(t) = \frac{1}{(1-t)^2}\frac{\sum_{k=0}^{N-1}\binom{N-1}{k}^2 t^{2k}}{(1-t^2)^{2(N-1)}} \tag{2.97}$$

Using one more mathematical fact, namely that the order of the pole of the Hilbert series equals the complex dimension of the space, we see that $\dim_{\mathbb{C}}\mathfrak{M}_{k=1,SU(N)} = 2N$ agrees with Eq. (2.27).

2.5.3 Generalization to other gauge groups

Even though we have to far restricted ourselves to considering $SU(N)$ instantons for matters of simplicity, one can easily generalize most of the above discussion. In particular, one can extend the ADHM construction presented above to any classical Lie group $SO(n)$ or $Sp(n)$ by modifying the gauge theory living on the D7 branes such that it obeys the respective gauge symmetry. Key to obtaining orthogonal or symplectic gauge groups is the introduction of O7 orbifold planes (see e.g. [18] for a review) and they have been studied extensively.

By an analogous treatment as above, one can show [7] for $k=1$ that the Hilbert series

Lie group G	Dynkin label of Adj_G^k	Dual Coxeter number $h_G^\vee$
$A_N = SU(N+1)$	$[k,0,\ldots,0,k]$	$N+1$
$B_{N\geq 3} = SO(2N+1)$	$[0,k,0,\ldots,0]$	$2N-1$
$C_{N\geq 2} = Sp(2N)$	$[2k,0,\ldots,0]$	$N+1$
$D_{N\geq 4} = SO(2N)$	$[0,k,0,\ldots,0]$	$2N-2$

Table 3: Dynkin labels and dual Coxeter numbers of the classical Lie groups

of the instanton moduli space for a classical Lie group $G \in \{A_N, B_N, C_N, D_N\}$ is given by

$$g_{k=1,G}^{\mathrm{Higgs}} = \frac{1}{(1-xt)\left(1-\frac{t}{x}\right)} \sum_{k=0}^{\infty} \mathrm{Adj}_G^k \, t^{2k} \;, \tag{2.98}$$

where Adj^k is the representation with Dynkin labels listed in Table 3. Furthermore one can reproduce another result known from index calculations, namely that

$$\dim \mathfrak{M}_{k=1,G} = 4h_G^\vee \;, \tag{2.99}$$

where $h_G^\vee$ is the dual Coxeter number of the Lie group G.

Lie group G	Dynkin label of Adj_G^k	Dual Coxeter number $h_G^\vee$
E_6	$[0,k,0,0,0,0]$	12
E_7	$[k,0,0,0,0,0,0]$	18
E_8	$[k,0,0,0,0,0,0,0]$	30
F_4	$[k,0,0,0]$	9
G_2	$[0,k]$	4

Table 4: Dynkin labels and dual Coxeter numbers of the exceptional Lie groups

For exceptional Lie groups, the situation becomes considerably more involved, as there is no known ADHM construction. In fact, even a Lagrangian description of a gauge theory exhibiting exceptional symmetries is lacking. For the case of $E_{6,7,8}$, theories with respective flavor symmetries arise from the low-energy effective action of 3, 4 and 6 M5-branes wrapping two-spheres with three punctures, the properties of which depend on the respective group [17, 5]. For the case of F_4 and G_2 not even such a description is known. Nevertheless, building on the systematic structure observed for classical Lie groups, the authors of [7] conjecture that the same pattern continues to hold, with the respective representations given by Table 4. To non-trivially check this conjecture, one can exploit certain dualities of the these strongly coupled SCFT's and we refer to the original paper for details.

3 Instanton effects in Quantum Field Theory

In partial constrast to the previous chapter, which focused on the study of instantons and their moduli spaces mainly for its own sake, this chapter will highlight some of the physical implications that instantons have. Even though the actual calculation of instanton effects has often been challenging, the inclusion of non-perturbative effects has greatly improved the qualitative understanding of modern quantum field theory.

Although we have already encountered similarities between instanton solutions in gauge theories and the quantum mechanical example of tunneling studied in the introduction, we start out by reconsidering our initial instanton set-up in order to make it clear that instantons can indeed be considered a tunneling effect. Having done that, we explore some rather straight-forward implications for the quantum theory. We then explain in more detail how to perform the semi-classical approximation of the path integral in an instanton background before finally concluding with a few comments on supersymmetric field theories.

Readers interested in more detail of the physical applications can consult for example [10] or [42]. A very recently published textbook covering the same ground is [37], which also contains some discussions of supersymmetry. For details on instanton calculus mainly with, but also without supersymmetry, we again refer to [12].

3.1 Tunneling interpretation of instantons

Maybe the first question one should ask in order to understand in what sense instantons are related to tunneling is which degree of freedom of the field theory can tunnel through a potential. Even though most degrees of freedom are confined by quadratic potentials, there exists one that is not. To see which degree that is, let us consider the Yang-Mills fields A_m^a where $a = 1, \ldots \dim SU(N)$ in temporal gauge, i.e. $A_4^a = 0$. Then the Hamiltonian is

$$H = \frac{1}{2} \int \mathrm{d}^3 x \, (E_i^a E_i^a + B_i^a B_i^a) \ , \tag{3.1}$$

where the "electric" fields E_i^a arise as conjugate momenta of the gauge field A_i^a and $i = 1, 2, 3$ only runs over spatial components. Since a gauge field only contains two degrees of freedom, the description in Eq. (3.1) in terms of three components is not completely gauged yet - gauge transformations independent of time x^4 are still unrestricted. Nevertheless, it is important not to impose an additional gauge condition, as we are to study different topological sectors of gauge transformations.

Just as in the previous chapter, we are interested in classical vacuum solutions and at any given time t,

$$A_m(x_i) = U(x_i)\partial_m U^{-1}(x_i) \tag{3.2}$$

is such a solution for $x_i \in \mathbb{R}^3$, as already stated in Eq. (2.6) with the slight restriction that $U \in SU(N)$ must now be independent of time. As we are interested in tunneling between different vacua, we restrict to such states that can be connected by finite actions. Hence one additionally demands that

$$U(x_i) \to \mathbb{1} \quad \text{as} \quad |x_i| \to \infty\ , \tag{3.3}$$

thereby compactifying $\mathbb{R}^3$ to S^3 by adding the additional point at infinity. Those constraints should be familiar from earlier on and one finds again that solutions to these two equations are classified by their homotopy class and therefore carry a charge $k \in \pi_3(S^3) \cong \mathbb{Z}$. In Eq. (2.10) we saw that one can obtain k by integration. Writing out the components, one has

$$k = \frac{1}{32\pi^2}\int \mathrm{d}^3x\, K^0\ , \tag{3.4}$$

where the *Chern-Simons-current*

$$K^m = 2\epsilon^{mnrs}\left(A_n^a\partial_r A_s^a + \frac{1}{3}f^{abc}A_n^a A_r^b A_s^c\right) \tag{3.5}$$

is the same as in Eq. (2.11), only that the boundary is now the spatial slice at a fixed time t.

While for $k \in \mathbb{Z}$ the field A is pure gauge, one can obtain any real k for field configurations

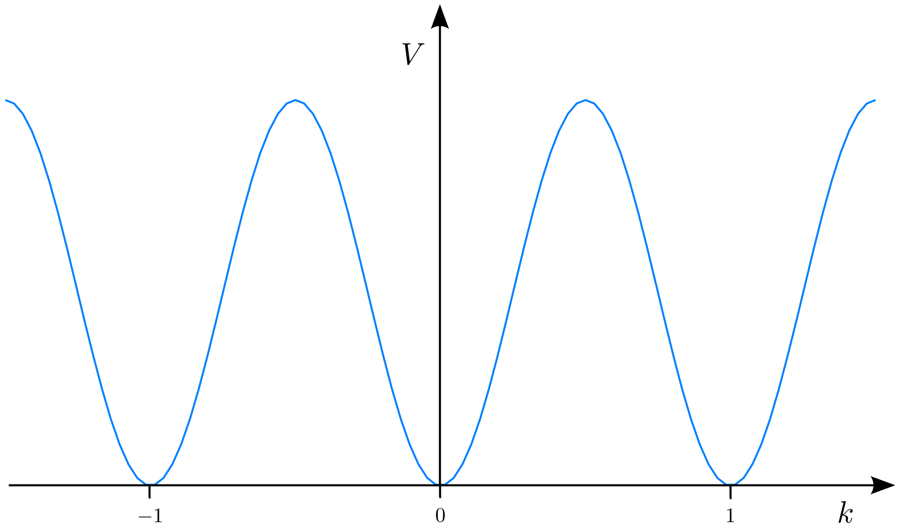

Figure 4: Schematic visualization of the periodic potential in k direction

with non-zero energy. At the same time, two field configurations whose charges k_1, k_2 satisfy $k_1 - k_2 \in \mathbb{Z}$ are related by a gauge transformation and therefore describe the same physical situation. Our gauge theory therefore has one physical direction with the topology of a circle and k is a coordinate on the covering space $\mathbb{R}$ of this circle S^1. One can therefore describe this situation by a potential that is periodic in k direction and has zeros for integer values of k, see Figure 4. Quantum mechanical tunneling occurs between different minima of the potential, i.e. one and the same point on S^1, but not on its covering space.

Consider now two field configurations at times t_- and t_+ with charges k_- and k_+ and take spacetime to be $[t_-, t_+] \times S^3$. The boundary of spacetime then just consists of two S^3 with opposite orientations, and, performing the integral in Eq. (2.11), we find an instanton charge of $k = k_+ - k_-$. Instantons can therefore be considered to be the least action field configurations connecting two degenerate vacua belonging to two different topological sectors.

Note that there are infinitely many classical vacua parametrized by an integer number. They are sometimes called *pre-vacua*.

3.1.1 Theta vacua

To extend this purely classical discussion to quantum mechanics, recall how to deal with a periodic system e.g. from condensed matter. In particular, Bloch's theorem states that a wavefunction on a circle with angular coordinate $2\pi \cdot k$ must obey

$$\Psi(k+1) = \mathrm{e}^{i\theta}\Psi(k) \tag{3.6}$$

with some fixed angle $\theta \in [0, 2\pi)$. Extending the analogy to quantum field theory by trading wave functions for wave functionals, one therefore finds that the quantum eigenstates must be

$$\Psi_\theta = \sum_{n\in\mathbb{Z}} \mathrm{e}^{in\theta}\Psi_n \ , \tag{3.7}$$

where Ψ_n are the wave functions corresponding to the classical pre-vacua. θ, the *vacuum angle* parametrizing the different quantum mechanical vacua, is a new and purely quantum parameter of the theory.

To incorporate θ into the Lagrangian description of quantum field theory, one must modify the action (2.4) to

$$S = -\frac{1}{2g^2}\int \mathrm{d}^4x\, \mathrm{Tr}\left(F_{mn}F^{mn}\right) - \frac{i\theta}{16\pi^2}\int \mathrm{d}^4x\ \mathrm{Tr}\left(F_{mn} \star F^{mn}\right) \ . \tag{3.8}$$

In the case of QCD, it is important to note that this additional term breaks both P and CP invariance unless θ is a multiple of π. Since P and CP violation of the strong interaction have to this date not been observed, there are strong experimental bounds on θ [24]:

$$\theta \leq 10^{-11} \tag{3.9}$$

There exist several proposals to avoid fine-tuning, among them the promotion of θ to be the expectation value of a new field, the *axion* [41, 43], relaxing to zero in its potential. So far, this hypothetical particle has not been detected and estimates [24] place it well above energies reachable in current accelerators.

3.1.2 Chiral anomaly

Moving away from considering pure Yang-Mills gauge theories, let us quickly mention some of the implications that the introduction of fermions can have. Consider the addition of a Dirac fermion, which results in an extra term

$$S_F = \int \mathrm{d}^4 x \bar{\psi}^A \left(-i\gamma^m \mathcal{D}_m - im\right) \psi_A \ , \tag{3.10}$$

where A is a flavour index running from 1 to N_f. Performing the path integral over Grassmannian variables ψ_A and $\bar{\psi}^A$, there is now an additional term proportional to

$$\det \left(-i\gamma^m \mathcal{D}_m - im\right)^{N_f} \ . \tag{3.11}$$

One can show [10] that $-i\gamma^m \mathcal{D}_m$ has only real eigenvalues λ_i. For every solution u_i with eigenvalue λ_i one can then define another solution $\bar{u}_i \equiv \gamma_5 u_i$ whose eigenvalue is $-\lambda_i$ as one sees directly using $\{\gamma_m, \gamma_5\} = 0$. Every non-zero eigenvalue λ_i therefore comes paired with another eigenvalue of the opposite sign. Writing the determinant as a product of eigenvalues, one has

$$\begin{aligned} \det \left(-i\gamma^m \mathcal{D}_m - im\right) &= \prod_n \left(\lambda_n + im\right) \\ &\sim m^l \prod_{n \text{ s.t. } \lambda_n > 0} \left(\lambda_n^2 + m^2\right) \ , \end{aligned} \tag{3.12}$$

where the proportionality constant is an irrelevant sign and l is the number of zero eigenvalues of $-i\gamma^m \mathcal{D}_m$, which remains to be determined. Since both summands in Eq. (3.12) are positive for $m > 0$, the addition of a massive Dirac fermion does not crucially change the discussion of the above section.

On the other hand, massless fermions could potentially lead to zero modes in the determinant and indeed, using the index theorem by Atiyah and Singer, one can show [10] that there are

$$l = k \tag{3.13}$$

zero modes in a background with instanton charge k. Does this zero mode mean that there are no instanton contributions in the case of massless fermions and we have to drop our tunneling interpretation altogether? It turns out that it does not.

In order to see why, recall that classically, massless fermions satisfy an additional symmetry, namely invariance under chiral rotations:

$$\psi_A \to \exp\left(i\alpha_A \gamma_5\right) \psi_A \tag{3.14}$$

Quantum mechanically, this symmetry is anomalous, meaning that although the action remains invariant under chiral transformations, the integration measure of the path integral does not. In fact, performing rotations as in Eq. (3.14) has the effect of shifting the vacuum angle [42]

$$\theta \to \theta + 2 \sum_{A=1}^{N_f} \alpha_A \ . \tag{3.15}$$

Since θ can be removed by a simple change of integration variables, it cannot be an observable anymore, confirming what we found above. However, instead of having a conserved axial current $\partial^m j_m^{\text{axial}} = 0$ associated with chiral transformations, the anomaly introduces a new term

$$\partial^m j_m^{\text{axial}} = \frac{N_f}{16\pi^2} F^a_{mn} \star F^{amn} \ . \tag{3.16}$$

Instantons are therefore responsible for violating the conservation of the axial current. Indeed, they can still be considered as a tunneling transition from one vacuum to another, but one also has to take into account that they change j_0^{axial}, the fermionic chiral charge density. Integrating

$$\int_{-\infty}^{\infty} \mathrm{d}t \int \mathrm{d}^3x\, \partial^0 j_0^{\mathrm{axial}} = \frac{N_f}{16\pi^2} \int \mathrm{d}^4x F^a_{mn} \star F^{amn} = 2kN_f \tag{3.17}$$

one finds that an instanton of charge k changes the chiral charge by $2kN_f$ units. An instanton in a theory with N_f flavours can therefore be interpreted as an object changing the chirality of $k \cdot N_f$ Weyl fermions, giving rise to the so-called *'t Hooft vertex* [38].

3.2 Semi-classical instanton corrections

Having recovered the tunneling interpretation of instantons, let us return to the starting point of our instanton discussion: The saddle point approximation of the gauge theory path integral. Becoming slightly more formal, we make use of the knowledge about instanton solutions gathered in the previous chapter to give a few more details on how to actually perform the semi-classical approximation around instanton solutions in Eq. (2.2). We largely follow [12].

To account for the newly justified θ-term, we introduce the complexified gauge coupling

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \tag{3.18}$$

and rewrite (2.2) in the quaternionic basis from Section 2.2.1 using Eq. (2.12).

$$\begin{aligned} S &= -2\pi i k\tau + \frac{1}{2}\int \mathrm{d}^4x_1 \mathrm{d}^4x_2\, \delta A_m(x_1) \frac{\delta^2 S}{\delta A_m(x_1)\delta A_n(x_2)} \delta A_n(x_2) + \dots \\ &= -2\pi i k\tau - \frac{1}{2}\int \mathrm{d}^4x\, \mathrm{Tr}\left(\delta \bar{A}^{\dot\alpha\alpha} \Delta^{(+)}{}_\alpha{}^\beta \delta A_{\beta\dot\alpha}\right) + \dots\ , \end{aligned} \tag{3.19}$$

We assumed that k is non-negative and defined the operator appearing in the quadratic variation of the action to be

$$\Delta^{(+)} \equiv -\not{\mathcal{D}}\bar{\not{\mathcal{D}}} = -\mathcal{D}^2 - F^{mn}\sigma_{mn}\ . \tag{3.20}$$

One must further include ghost fields arising from the gauge-fixing

$$S_{\text{ghosts}} = 2 \int d^4x \, \text{Tr} \left(b\mathcal{D}^2 c \right) \ . \tag{3.21}$$

In order to have a well-defined functional determinant, we split up δA_n into zero modes $\delta_\mu A_n$ and non-zero modes $\tilde{A}_n$ of $\Delta^{(+)}$

$$\delta A_n = \sum_\mu \zeta^\mu \delta_\mu A_n + \tilde{A}_n \ , \tag{3.22}$$

where as before Greek indices run from 1 to $\dim \mathfrak{M}_k$. The zero modes are orthogonal to the non-zero modes, and therefore it is straightforward to split up the functional integration. Howecer, the zero modes need not form an orthonormal basis with respect to the metric on the instanton moduli space and therefore we must include a factor of $\sqrt{g}$:

$$\int [dA_n] = \int \left(\sqrt{\det g(X)} \prod_\mu \frac{d\zeta^\mu}{\sqrt{2\pi}} \right) [d\tilde{A}_n] \tag{3.23}$$

In this formula, X are coordinates on the instanton moduli space and $g(X)$ is the respective metric. The spectrum of $\Delta^{(+)}$ is now suitably split up and, at least in principle, one could perform the usual Gaussian path integral over the remaining functions:

$$\int [db][dc][d\tilde{A}_n] \ \exp \left(\frac{1}{2} \text{Tr} \int d^4x \tilde{\bar{A}}^{\dot\alpha\alpha} {\Delta^{(+)}}_\alpha{}^\beta \tilde{A}_{\beta\dot\alpha} - 4b\mathcal{D}^2 c \right) = \frac{\det \left(-\mathcal{D}^2\right)}{\det' \left(\Delta^{(+)}\right)} \tag{3.24}$$

Here the prime on the determinant indicates that zero modes are to be excluded when evaluating the operator in the instanton background.

Instead of integrating over expansion coefficients ζ^μ it would be preferrable to integrate over the moduli space itself. Using an insertion of unity reminiscent of the Fadeev-Popov gauge fixing trick, i.e.

$$1 \equiv \prod_\mu dX^\mu \left| \det \frac{\partial \zeta^\sigma g_{\sigma\nu}}{\partial X^\mu} \right| \prod_\nu \delta \left(\zeta^\sigma g_{\sigma\nu}(X) \right) \ , \tag{3.25}$$

one can show [12] that to leading order in the coupling constant g one has

$$\int [\mathrm{d}b][\mathrm{d}c][\mathrm{d}A_n]\Big|_k \, \mathrm{e}^{-S[A_n,b,c]} = \mathrm{e}^{2\pi i k\tau} \int_{\mathfrak{M}_k} \omega \, \frac{\det\left(-\mathcal{D}^2\right)}{\det'\left(\Delta^{(+)}\right)} + \mathcal{O}(g) \tag{3.26}$$

where ω is the canonical volume form of the Hyper-Kähler space $\mathfrak{M}_k$

$$\int_{\mathfrak{M}_k} \omega \equiv \int \sqrt{\det g(X)} \prod_{\mu} \frac{\mathrm{d}X^\mu}{\sqrt{2\pi}} \, . \tag{3.27}$$

Having derived $\mathfrak{M}_k$ explicitly as a Hyper-Kähler quotient in the previous chapter, one can determine ω in a straightforward fashion. Nevertheless, since we will not need it for the remaining discussion we once again refer to [12] for a concrete formula and close with a more qualitative consideration instead. Before doing so, however, note that both operators appearing in the functional determinants must be evaluated in the respective instanton backgrounds and will therefore depend non-trivially on the collective coordinates X^μ. Although the determinant factor can be computed from ADHM data [33] its calculation is tedious. Fortunately, supersymmetry simplifies the discussion greatly: In a supersymmetry theory the fluctuation determinant is an irrelevant constant [38].

3.2.1 The clustering limit

Even though most of our discussion treated the instanton moduli spaces $\mathfrak{M}_k$ associated to different charges on an equal footing, we only saw concrete solutions for the case of $k = 1$. Hence one naturally wonders how to interpret solutions with higher charge and how to relate them to the BPST instanton. It turns out that an instanton of charge k can roughly be thought of k charge 1 instantons. In the limit of large separations of the individual BPST instantons, the *clustering limit*, the instanton solution with charge k is then the superposition of the different BPST instantons. Similarly, the instanton measure on $\mathfrak{M}_k$ must split up into k integrations over $\mathfrak{M}_1$

$$\int_{\mathfrak{M}_k} \omega \xrightarrow{\text{clustering}} \frac{1}{k!} \int_{\mathfrak{M}_1} \omega \times \ldots \times \int_{\mathfrak{M}_1} \omega \, . \tag{3.28}$$

Since calculating the Hilbert series of the instanton moduli space increases very quickly in difficulty for higher k, one can in principle try to use the same approximation there, namely that

$$\mathfrak{M}_k \xrightarrow{\text{clustering}} \mathrm{Sym}^k\, \mathfrak{M}_1 \tag{3.29}$$

in the clustering limit. In this limit the Hilbert series can therefore be approximated by

$$\begin{aligned} g^{\text{Higgs}}_{k,G}(t;x;\{z\}) &= \mathrm{Sym}^k g^{\text{Higgs}}_{1,G}(t;x;\{z\}) \\ &= \sum_{\substack{m_i \text{ s.t.} \\ \sum_i i m_i = k}} \prod_i \frac{f(t^i;x^i;\{z^i\})^{m_i}}{i^{m_i}\cdot m_i!} \,. \end{aligned} \tag{3.30}$$

In the simplest case of $k = 2$ the above formula only contains two summands

$$g^{\text{Higgs}}_{2,G}(t) = \frac{1}{2}\left[\left(g^{\text{Higgs}}_{1,G}(t)\right)^2 + g^{\text{Higgs}}_{1,G}(t^2)\right] \tag{3.31}$$

and, using that $g^{\text{Higgs}}_{k,G}$ has a reducible component $\frac{1}{(1-xt)(1-\frac{t}{x})}$ for all values of k, one finds that the two summands contribute as residues at $x = \pm t$. This statement generalizes to higher instanton charges for which one finds poles at $x = \mathrm{e}^{i\alpha_l}t$ for finitely many α_l [20]. Although only approximate, this approach provides a convenient way of making consistency checks.

3.3 Instantons and supersymmetry

In the last part of this chapter, we touch upon a few properties of instantons in supersymmetric gauge theories. Having seen in the last section that instanton calculus can simplify tremendously in the presence of supersymmetry, we discover that the inclusion of both scalar and fermionic fields can at the same time lead to extra complications. Last of all, we take a short look at the chiral $U(1)$ R-symmetry of gauge theories with 8 supercharges and find again that it is anomalous.

Please be reminded that there are plenty more applications one could review, among them the lifting of four-dimensional instanton solutions in supersymmetric theories to extended objects in higher dimensions, e.g. instantons in $\mathcal{N} = 4$ Super-Yang-Mills theory to the related theory with $\mathcal{N} = 1$ in ten dimensions, in which they appear as branes. Viewing the same problem in different theories can give valuable insights and simplify calculations.

Before treating the most general supersymmetric case, let us extend the discussion of fermions and consider the addition of a adjoint Weyl fermion instead of a fundamental Dirac fermion to the usual $SU(N)$ Yang-Mills gauge theory. This scenario corresponds to the simplest $\mathcal{N} = 1$ Super-Yang-Mills (SYM) theory without additional matter. We again try to minimize the action by setting the gauge field to its ADHM value before looking for zero modes of the Dirac operator

$$\bar{\not{\mathcal{D}}}\lambda^A = 0 \quad \text{and} \quad \not{\mathcal{D}}\bar{\lambda}_A = 0 \,. \tag{3.32}$$

Here the covariant derivatives are evaluated in the adjoint representation with an instanton background of charge k. Using once again the Atiyah-Singer index theorem, one can show that for $k \geq 0$ $\not{\mathcal{D}}$ is a positive definite operator, but $\bar{\not{\mathcal{D}}}$ is not. In fact, we encountered the same equation before in (2.25) and, since there is no open index this time, know that there must be exactly $2kN$ zero modes. One calls the solutions to Eq. (3.32) *fermion zero modes* of the instanton solution and the whole solution satisfying the full equations of motion is sometimes referred to as a *super-instanton*.

Denoting the linearly independent solutions to Eq. (3.32) by $\Lambda_{\alpha,i}$, the most general fermionic solution reads

$$\lambda_\alpha = \sum_{i=1}^{2kN} \psi^i \lambda_{\alpha,i} \,, \qquad \bar{\lambda}_{\dot{\alpha}} = 0 \,, \tag{3.33}$$

where ψ^i are Grassmann variables. Just as the bosonic collective coordinates can be interpreted as coordinates on the instanton moduli space $\mathfrak{M}_k$, the fermionic collective coordinates can be associated with *symplectic tangent vectors* of $\mathfrak{M}_k$ due to its Hyper-Kähler structure.

3.3.1 Supersymmetry and Quasi-Instantons

Progressing either to the case of more than 4 supercharges or the addition of matter to the $\mathcal{N} = 1$ theory, it becomes necessary to include scalar fields ϕ with Yukawa couplings $\sim \bar{\lambda}\phi\lambda$ to fermions. As above, our approach is to set to gauge fields and fermions to the same values they would have in a setting without scalars. The new equation for the scalars is the usual Laplace equation with a source term bilinear in the fermions. If scalars acquire non-zero VEV's, then suddenly they "backreact": Unlike in the purely fermionic case above, the scalars modify the equations of motion both for the gauge fields and the fermions and invalidate the original ADHM solutions. At least for the BPST instanton, there is an intuitive way of understanding this behaviour. By introducing scalar VEVs the theory gains a mass scale and hence loses its conformal invariance. Without scale invariance, we would not expect the size ρ to be a free parameter of our solution anymore.
To make matters worse, scalars can acquire bilinear Grassmann components from the Yukawa couplings, thereby modifying the equations of motion for A_n and λ even in the absence of "non-Grassmann" VEVs.

Generically, these corrections occur whenever they are not forbidden by symmetries of the theory, as is example the case for $\mathcal{N} = 2$ SYM without scalar VEVs. Nevertheless, not all hope is lost. It turns out that the original solutions still solve the instanton equations to lowest order in the coupling constant g and is therefore called a *quasi-instanton.* It can be described by an effective action with the corrections appearing as interaction terms, which reads in the case of $\mathcal{N} = 4$ SYM

$$S_{\text{effective}} = \frac{8\pi^2 k}{g^2} - ik\theta + \frac{\epsilon_{ABCD}}{96} R_{ijkl}\psi^{iA}\psi^{jB}\psi^{kC}\psi^{lD} \ . \tag{3.34}$$

Here, R_{ijkl} is the curvature tensor on the instanton moduli space and depends on the bosonic collective coordinates X^μ.

In order to account for scalar VEVs, Affleck developed a formalism and called the corresponding quasi-instantons *constrained instantons.* Roughly speaking, the effective action for the instanton contains a potential whose minima correspond to sending the instanton size to zero, e.g.

$$S_{\text{effective}} = \frac{8\pi^2}{g^2} - i\theta + 4\pi^2\rho^2 \langle \mathrm{Tr}\left(\phi^2\right)\rangle \tag{3.35}$$

for the simplest case of $SU(2)$ with an adjoint scalar ϕ. The integration over the instanton moduli space in Eq. (3.26) is then modified:

$$\int_{\mathfrak{M}_k} \omega \to \int_{\mathfrak{M}_k} \omega \, \mathrm{e}^{-S_{\text{effective}}} \tag{3.36}$$

Much more detail on constrained instantons can be found in Affleck's original paper [1].

3.3.2 Anomalies in the abelian R-symmetry

To conclude this chapter, we include another short application of instanton calculus very similar to the one of Section 3.1.2. As above we encounter a chiral anomaly due to instanton effects, but this time we use a simple counting argument based on zero mode counting in order to determine the anomalous term that we could equally well have used before, too.

Field	$SU(2)_R \times U(1)_R$	$U(1)_J \times U(1)_R$
A_n	$[0]_0$	$(0;0)$
λ	$[1]_1$	$(1;1)$
ψ		$(-1;1)$
ϕ	$[0]_2$	$(0;2)$

Table 5: Field content of $\mathcal{N}=2$ vector multiplet and its charges

We are interested in the anomalies of the $SU(2)\times U(1)$ R-symmetry of $\mathcal{N}=2$ SYM-theory and restrict to pure gauge theory without matter multiplets. The field content of the respective vector multiplet is listed in Table 5, where $U(1)_J$ is the diagonal subgroup of $SU(2)_R$. All the fields also transform under the adjoint representation of the gauge group G. The key point is that after combining the two Weyl fermions into one four-component Dirac spinor

$$\psi_D = \begin{pmatrix} \lambda_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \tag{3.37}$$

the transformation under $U(1)_J \times U(1)_R$ can be written as

$$\psi_D \xrightarrow{U(1)_J} \mathrm{e}^{i\alpha}\psi_D \qquad \text{and} \qquad \psi_D \xrightarrow{U(1)_R} \mathrm{e}^{i\alpha\gamma_5}\psi_D \ . \tag{3.38}$$

Therefore $U(1)_R$ is a chiral symmetry prone to anomalies. To determine the first fermionic correlator with instanton corrections, recall that there are $2N$ fermionic zero modes for every left-handed fermion in the adjoint representation of $SU(N)$. With every fermion zero mode comes and integration over Grassmann variables ψ^i and, since

$$\int \mathrm{d}\psi^i = 0 \qquad \text{and} \qquad \int \mathrm{d}\psi^i\, \psi^j = \delta^{ij} \tag{3.39}$$

only correlators with as many fermionic insertions as there are zero modes will contribute. The first non-vanishing correlator is therefore

$$G^{(4N)} = \langle \lambda(x_1)\ldots\lambda(x_{2N})\psi(x_{2N+1})\ldots\psi(x_{4N})\rangle \ , \tag{3.40}$$

which transforms under $U(1)_R$ as

$$G^{(4N)} \to \mathrm{e}^{i4N\alpha}\cdot G^{(4N)} \qquad \text{with } \alpha \in [0, 2\pi) \ . \tag{3.41}$$

Since $G^{(4N)}$ remains invariant only for $\alpha = \frac{n\pi}{2N}$, $n = 1, \ldots, 4N$, instanton effects break $U(1)_R$ to the discrete subgroup $\mathbb{Z}_{4N}$. On the other hand, the center $\mathbb{Z}_2 \subset SU(2)$ acts as $(\lambda, \psi) \to -(\lambda, \psi)$ and is therefore already contained in $\mathbb{Z}_{4N}$. The true R-symmetry group of $\mathcal{N} = 2$ SYM-theory is hence

$$\frac{SU(2) \times \mathbb{Z}_{4N}}{\mathbb{Z}_2} \ . \tag{3.42}$$

There is one more restriction: The operator $\mathrm{Tr}\,\phi^2$ has charge 4 under $U(1)_R$ and for non-zero vacuum expectation values of $\mathrm{Tr}\,\psi^2$ $\mathbb{Z}_{4N}$ breaks to $\mathbb{Z}_4$. In this case one ends up with

$$\frac{SU(2) \times \mathbb{Z}_4}{\mathbb{Z}_2} \cong SU(2) \times \mathbb{Z}_2 \tag{3.43}$$

with the $\mathbb{Z}_2$ acting as $\operatorname{Tr} \phi^2 \to -\operatorname{Tr} \phi^2$.

4 The Seiberg-Witten approach to $\mathcal{N} = 2$ SUSY

In this chapter, we present the approach to obtaining the low-energy effective action of a pure $\mathcal{N} = 2$ Super-Yang-Mills theory discovered by Seiberg and Witten [34]. Even though the introduction of the so-called *Seiberg-Witten curve*, which we will explain shortly, made a generalization of their first result to higher-rank gauge groups and the inclusion of matter [35] relatively straight-forward, we limit ourselves to the simplest possible case.

Compared to the previous chapters, this one differs in quite a few aspects. Most importantly, the majority of the machinery developed above will not be of any use to us. Apart from some general knowledge about instantons, such as the general structure of instanton contributions and their role in the anomaly of the $U(1)_R$ symmetry, we do not need instanton techniques. In particular, we will not use the semi-classical approximation of the path integral outlined in Section 3.2, as the analysis by the above two authors relies mainly on global properties of the theory. Despite all of that, we believe that this chapter fits in nicely with the rest, as it is a rare example of an interacting field theory whose instanton corrections can be determined to all orders.

Apart from the original papers, there exist several reviews of the topic, e.g. [2] or [26] and in the following we will borrow from all of them.

4.1 The set-up for pure $SU(2)$ SYM

Let us now explain the concrete set-up. For the time being, we revert back to Minkowski spacetime in order to follow the conventions of [34, 2]. The problem at hand is the following: Consider pure $\mathcal{N} = 2$ Super-Yang-Mills theory with gauge group $G = SU(2)$. Then the field content of the theory is an $\mathcal{N} = 2$ vector multiplet containing a vector field A_μ, a complex scalar ϕ and two fermionic fields λ and ψ, all of which transform in the adjoint representation of $SU(2)$ (see also Table 5). In $\mathcal{N} = 1$ language, these fields can be packaged into a vector multiplet and a chiral multiplet for which the Lagrangian density is given by

$$\mathcal{L} = \frac{1}{8\pi}\Im\,\mathrm{Tr}\left[\tau\left(\int \mathrm{d}^2\,\theta W^\alpha W_\alpha + 2\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\,\varPhi^\dagger \mathrm{e}^{2V}\varPhi\right)\right] \ . \tag{4.1}$$

Inserting the component expressions for the superfields W^α and $\varPhi$, this yields the scalar potential

$$V = \frac{1}{2g^2}\,\mathrm{Tr}\left([\phi,\phi^\dagger]^2\right) \ , \tag{4.2}$$

which is minimized if and only if ϕ takes values in the Cartan subalgebra $H \subset SU(2)$. Without loss of generality, one can take ϕ to be

$$\phi = a\sigma_3 = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \ , \tag{4.3}$$

since all other choices of a Cartan subalgebra can be related to this one by a gauge transformation. To eliminate the remaining redundancy under the gauge transformation $a \to -a$ one chooses the Casimir operator of $SU(2)$,

$$u = \frac{1}{2}\langle \mathrm{Tr}\,\phi^2\rangle \ , \tag{4.4}$$

as a coordinate on $\mathcal{M}$, the space of vacua. Classically, $u = \frac{1}{2}a^2$ and one observes that for generic values of u, the gauge group $SU(2)$ is broken down to $U(1)$ and the gauge bosons $W^\pm$ corresponding to linear combinations of A^1 and A^2 become massive. In the absence of quantum effects, there is exactly one singular point, $u = 0$, at which the full gauge symmetry is restored.

To determine the true quantum low-energy effective action, we take a Wilsonian approach. For renormalization scales μ much larger than the symmetry breaking scale a, we have an asymptotically free $SU(2)$ theory with negative β-function, while for $\mu \ll a$ the $W^\pm$ bosons are integrated out and effectively one obtains a $U(1)$ theory with vanishing β-function. We set $\mu = a$ and search for the gauge coupling of the effective theory for any value of u.

Due to the asymptotic freedom of the $SU(2)$ theory and the fact that the gauge coupling "freezes out" at the symmetry breaking scale a, we can expect the theory to be weakly coupled and semi-classical methods to work only in the region $\frac{u}{\Lambda^2} \to \infty$. It is the achievement of Seiberg and Witten to nevertheless determine the effective gauge coupling for all values of u and we review their argument.

4.1.1 General form of the action

A general result from supersymmetry states that the low-energy action for $\mathcal{N}=2$ $U(1)$ gauge theory is completely determined by a preprotential $\mathcal{F}(A)$, which is a holomorphic function of the chiral superfield A. Given $\mathcal{F}$, the Lagrangian density takes the form

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\Im\left[\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, \frac{\partial \mathcal{F}(A)}{\partial A}\bar{A} + \int \mathrm{d}^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial A^2} W^\alpha W_\alpha\right] , \tag{4.5}$$

where $\bar{A}$ is the conjugate of A instead of a classical solution to the equations of motion as in Chapter 2. Comparison with Eq. (4.1) shows that the complexified gauge coupling is given by

$$\tau(a) = \frac{\partial^2 \mathcal{F}(A)}{\partial A^2} , \tag{4.6}$$

where a is as above the scalar component of the chiral multiplet. The metric on the field space can then be written as

$$\mathrm{d}s^2 = \tau(a)\mathrm{d}a\mathrm{d}\bar{a} . \tag{4.7}$$

On general grounds, the prepotential will be

$$\mathcal{F} = \mathcal{F}_{\text{pert}} + \mathcal{F}_{\text{non-pert}} , \tag{4.8}$$

while classically one would expect to have $\mathcal{F} = \frac{1}{2}\tau_0 A^2$ with τ_0 the bare coupling. To determine the one loop contribution one simply integrates the β-function of the theory, which is $\beta(g) = \frac{-g^3}{4\pi^2}$. Using the scale at which g "freezes out" as integration limit, one finds

$$\mathcal{F}_{\text{pert}} = \frac{i}{2\pi} A^2 \log\left(\frac{A^2}{\Lambda^2}\right) . \tag{4.9}$$

Making use of $\mathcal{N} = 2$ supersymmetry, it can then be shown [36] that $\mathcal{F}$ does not receive higher perturbative corrections. On the other hand, non-perturbative corrections can occur and will be proportional to $\mathrm{e}^{-8\pi^2 k/g^2}$. Using the β-function again, this can be rewritten as

$$\mathrm{e}^{-8\pi^2 k/g^2} = \left(\frac{\Lambda}{A}\right)^{4k} . \tag{4.10}$$

Finally, one argues that the anomalous $U(1)_R$ symmetry is restored if one assigns Λ a charge of two. In order for $\mathcal{F}$ to transform homogeneously, there must be an additional factor of A^2 in the non-perturbative correction.

Putting everything together, one therefore arrives at the following expression for the prepotential:

$$\mathcal{F} = \frac{i}{2\pi} A^2 \log\left(\frac{A^2}{\Lambda^2}\right) + A^2 \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{A}\right)^{4k} \tag{4.11}$$

As a closing remark, note that $\Im\tau(a) = \frac{\partial^2 \mathcal{F}}{\partial A^2}$ is a harmonic function and, as such, it has a global minimum if and only if it is constant. Discarding that possibility, one finds that $\tau(a)$ cannot be well-defined globally, but only on a local patch. We must therefore look for alternative descriptions of our theory and in the next section we do so.

4.1.2 Duality

For classical electromagnetism, it has long been known that after introducing a magnetic current, the equations of motion exhibit a $\mathbb{Z}_2$ symmetry, the so-called electromagnetic duality, swapping electric and magnetic sources and fields.

$$\partial_\mu \star F^{\mu\nu} = j^\nu_{\text{el}} , \qquad \partial_\mu F^{\mu\nu} = j^\nu_{\text{mag}} \tag{4.12}$$

Here the duality acts as $F^{\mu\nu} \to \star F^{\mu\nu}$ and $\star F^{\mu\nu} \to -F^{\mu\nu}$ accompanied by $j^{\nu}_{\text{el}} \to -j^{\nu}_{\text{mag}}$ and $j^{\nu}_{\text{mag}} \to j^{\nu}_{\text{el}}$. Note that the change to Minkowski spacetime implies that now $\star\star F^{\mu\nu} = -F^{\mu\nu}$.

We now implement the same duality at the level of a path integral and extend it to an abelian superfield. Let us start with an electric description $F = \mathrm{d}A$ without magnetic sources, or more precisely its superfield generalization $W = \mathcal{D}V$. To be able to treat W_α as an independent chiral field, we must introduce a Lagrange multiplier V_D enforcing the Bianchi identity $\mathcal{D}W = 0$. The kinetic term

$$\frac{1}{8\pi}\Im \int \mathrm{d}^2\theta \, \tau(A) W^\alpha W_\alpha \tag{4.13}$$

is therefore accompanied by

$$\frac{1}{4\pi}\Im \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, V_D \mathcal{D}W = -\frac{1}{4\pi}\Im \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, W^\alpha_D W_\alpha \; , \tag{4.14}$$

where $W_D = \mathcal{D}V_D$. Since these terms are only quadratic, one can perform the path integral over W_α exactly and obtains an equivalent kinetic term

$$\frac{1}{8\pi}\Im \int \mathrm{d}^2\theta \, \frac{-1}{\tau(A)} W^\alpha_D W_{D\alpha} \; . \tag{4.15}$$

Having rewritten the kinetic term for the gauge field, let us focus on the chiral Higgs field A and find its magnetic dual A_D. Following conventions and defining $h(A) = \frac{\partial \mathcal{F}(A)}{\partial A}$, the term for the chiral field reads

$$\Im \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, h(A)\bar{A} \; . \tag{4.16}$$

By simply setting $A_D = h(A)$, this term becomes

$$\Im \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, h_D(A_D)\bar{A_D} \; , \tag{4.17}$$

with $h_D(A_D) = h_D(h_A) = -A$. Using further that

$$\frac{-1}{\tau(A)} = \frac{-1}{\frac{\mathrm{d}h(A)}{\mathrm{d}A}} = -\frac{\mathrm{d}A}{\mathrm{d}h(A)} = h'_D(A_D) = \tau_D(A_D)\ , \tag{4.18}$$

finally yields the same Lagrangian as in Eq. (4.5) with A and τ replaced by A_D and τ_D.

Summarizing, we have found that we can express the original Lagrangian coupling electrically to a matter field A by an equivalent description coupling magnetically to A_D. At the same time, the coupling constant τ is transformed to τ_D via

$$\tau_D = \frac{-1}{\tau}\ . \tag{4.19}$$

Note further that the action is also invariant under a shift $\tau \to \tau + 1$, or equivalently $\theta \to \theta + 2\pi$. Since these two maps generate the group $SL(2,\mathbb{Z})$, we find that there are infinitely many equivalent descriptions of the same physical theory, all of which are related by some $SL(2,\mathbb{Z})$ map. These transformations act on the coupling constant as

$$\tau \to \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \quad \text{where} \quad \alpha\delta - \beta\gamma = 1 \quad \text{and} \quad \alpha, \beta, \gamma, \delta \in \mathbb{Z} \tag{4.20}$$

and on the fields via

$$\begin{pmatrix} A_D \\ A \end{pmatrix} \to \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} A_D \\ A \end{pmatrix}\ . \tag{4.21}$$

In order to make this $SL(2,\mathbb{Z})$-symmetry more manifest, it is helpful to find a suitable mathematical description of the variables $a_D(u)$ and $a(u)$. They are, of course, functions of the coordinate u parametrizing the moduli space $\mathcal{M}$. Take $X \cong \mathbb{C}^2$ to be the space in which (a_D, a) takes values, then $f : \mathcal{M} \to X$ with $f = (a_D(u), a(u))$.

X can further be endowed with a symplectic form $\omega = \Im \mathrm{d}a_D \wedge \mathrm{d}\bar{a}$. Since any symplectic form written in local coordinates as

$$\omega = \frac{i}{2} \sum_{j,k} h_{j\bar{k}} \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^{\bar{k}} \tag{4.22}$$

defines a Kähler metric $h_{j\bar{k}}$, so does our ω. Using the map f, we can therefore use the pull-back $f^*(\omega)$ as a natural metric on $\mathcal{M}$. Written in coordinates, on has

$$\begin{aligned} \mathrm{d}s^2 = \Im \frac{\mathrm{d}a_D}{\mathrm{d}u} \frac{\mathrm{d}\bar{a}}{\mathrm{d}\bar{u}} \mathrm{d}u \mathrm{d}\bar{u} &= -\frac{i}{2} \left(\frac{\mathrm{d}a_D}{\mathrm{d}u} \frac{\mathrm{d}\bar{a}}{\mathrm{d}\bar{u}} - \frac{\mathrm{d}\bar{a}_D}{\mathrm{d}\bar{u}} \frac{\mathrm{d}a}{\mathrm{d}u} \right) \mathrm{d}u \mathrm{d}\bar{u} \\ &= -\frac{i}{2} \epsilon_{\alpha\beta} \frac{\mathrm{d}a^\alpha}{\mathrm{d}u} \frac{\mathrm{d}\bar{a}^\beta}{\mathrm{d}\bar{u}} \mathrm{d}u \mathrm{d}\bar{u} \; , \end{aligned} \tag{4.23}$$

where in the final expression $a^\alpha = (a_D, a)$ and the invariant tensor of $SL(2, \mathbb{Z})$ with $\epsilon_{12} = 1$ makes the invariance that we found above obvious.

Hence, the appropriate mathematical structure describing the triplet a, a_D and u is that of an $SL(2, \mathbb{Z})$-bundle over the base space $\mathcal{M}$, which is parametrized by u. (a_D, a) is a section of this bundle taking values in its fiber X.

4.1.3 Central charges

Due to the constraints imposed by supersymmetry, there is a useful formula one can use in order to obtain lower bounds for the masses of the fields appearing in our theory. Although this is usually taught in standard supersymmetry courses, we quickly repeat the argument. For $\mathcal{N} = 2$ supersymmetry, the supercharges Q^a_α and $\bar{Q}_{\dot{\beta}b}$ satisfy

$$\{Q^a_\alpha, \bar{Q}_{\dot{\beta}b}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^a_b \tag{4.24a}$$

$$\{Q^a_\alpha, Q^b_\beta\} = 2\sqrt{2} \epsilon_{\alpha\beta} \epsilon_{ab} Z \tag{4.24b}$$

$$\{\bar{Q}_{\dot{\alpha}a}, \bar{Q}_{\dot{\beta}b}\} = 2\sqrt{2} \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{ab} Z \; , \tag{4.24c}$$

where the *central charge* Z is a constant operator taken to commute with all other generators. Q and $\bar{Q}$ can be linearly combined into operators

$$a_\alpha = \frac{1}{2} \left(Q^1_\alpha + \epsilon_{\alpha\beta} (Q^2_\beta)^\dagger \right) \quad \text{and} \quad b_\alpha = \frac{1}{2} \left(Q^1_\alpha - \epsilon_{\alpha\beta} (Q^2_\beta)^\dagger \right) \tag{4.25}$$

satisfying the fermionic Clifford algebrae

$$\{a_\alpha, a^\dagger_\beta\} = \delta_{\alpha\beta} \left(M + \sqrt{2} Z \right) \quad \text{and} \quad \{b_\alpha, b^\dagger_\beta\} = \delta_{\alpha\beta} \left(M - \sqrt{2} Z \right) \; . \tag{4.26}$$

Here we chose the rest frame for a massive particle to be such that $P_\mu = (M, 0, 0, 0)$. Since

$$\langle\psi|\{b_\alpha, b_\alpha^\dagger\}|\psi\rangle = |b_\alpha |\psi\rangle |^2 + |b_\alpha^\dagger |\psi\rangle |^2 \geq 0 \quad \forall\psi \in \mathcal{H} \tag{4.27}$$

$\{b_\alpha, b_\alpha^\dagger\}$ is a positive operator. Therefore the mass of the multiplet is bounded by

$$M \geq \sqrt{2}Z \; , \tag{4.28}$$

which is known as the BPS bound. In particular, fields saturating this bound correspond to lower-dimensional representations, so-called short multiplets, which are annihilated by b_α. Since these fields lie in different representations than those with higher masses, small perturbations will not change this structure and their masses are therfore protected from quantum corrections.

Determining the value of the central charge is straight-forward, but lengthy and we refer to [2] for details. The main idea is that the charges Q appearing on the left hand side of Eq. (4.24) are given by the space integral over the time component of the supercurrents $S_\alpha^{a\mu}$ associated with the supersymmetry transformations, i.e.

$$Q_\alpha^a = \int \mathrm{d}^3x \, S_\alpha^{a\mu} \tag{4.29}$$

Going through all the algebra, one finds that the resulting integral is that over a total derivative and therefore only boundary terms contribute. Evaluating those for the effective action of Eq. (4.5) then yields

$$Z = an_e + a_D n_m \tag{4.30}$$

for a particle of electric charge n_e and magnetic charge n_m, where a is the vacuum expectation value of the electrically charged Higgs field and a_D that of its magnetic dual.

With the knowledge of the central charge of our $\mathcal{N} = 2$ gauge theory, there is one more observation to be made. As we saw in the previous section, the chiral fields (A_D, A) transform under a duality transformation $M \in SL(2,\mathbb{Z})$ as in Eq. (4.21) and therefore so do their scalar components $a^\alpha = (a_D, a)$. At the same time, a physically equivalent description must have the same observable masses and therefore the charges $n_\alpha = (n_m, n_e)$ of the fields must transform as

$$n \to n \cdot M^{-1} \qquad M \in SL(2,\mathbb{Z}) \ . \tag{4.31}$$

4.2 Monodromies

Up to this point, we have explored different ways of formulating one and the same theory in terms of different sets of fields. In doing so, we discovered that all these descriptions are related by $SL(2,\mathbb{Z})$ transformations and we saw evidence that any such description can only be valid in a local patch of the whole moduli space $\mathcal{M}$. Nevertheless, we have not made much tangible progress in exploring $\mathcal{M}$, since the only region under control is that satisfying $\frac{u}{\Lambda^2} \to \infty$. In this section, we will present the physical arguments given by Seiberg and Witten to derive similar statements about other patches of $\mathcal{M}$, using $SL(2,\mathbb{Z})$ transformations to rewrite the theory in terms of variables in which it is weakly coupled. Fortunately, and somewhat miraculously, it turns out that - up to irrelevant ambiguities - there is a unique way of patching together these different local patches and we thereby control of all regions of $\mathcal{M}$.

Let us begin by a reinspection of the patch $\frac{u}{\Lambda^2} \to \infty$ for which we obtain an asymptotically free theory with $u = \frac{1}{2}a^2$. In this regime, non-perturbative corrections to the prepotential are supressed and Eq. (4.11) simplifies to

$$\mathcal{F}(a) = \frac{i}{2\pi} a^2 \log\left(\frac{a^2}{\Lambda^2}\right) \ , \tag{4.32}$$

where we exchanged the chiral field A by its scalar component a. Since $a_D = \frac{\partial \mathcal{F}(a)}{\partial a}$, we also have

$$a_D = \frac{ia}{\pi} \log\left(\frac{a^2}{\Lambda^2}\right) + \frac{ia}{\pi} \ . \tag{4.33}$$

Consider now a closed loop on the u-plane around the origin $u = 0$, i.e. $u \to e^{2\pi i} \cdot u$. Then we have the transformation

$$a \to -a \tag{4.34}$$

therefore $\log(a) \to \log(a) + i\pi$. From Eq. (4.33) one has that a_D transforms as

$$\begin{aligned} a_D &\to -\frac{2ia}{\pi}\left(\log\left(\frac{a}{\Lambda}\right) + i\pi\right) - \frac{ia}{\pi} \\ &= -\frac{2ia}{\pi}\log\left(\frac{a}{\Lambda}\right) - \frac{ia}{\pi} + 2a \\ &= -a_D + 2a \; . \end{aligned} \tag{4.35}$$

In compact form, (a_D, a) transforms as $(a_D, a)^T \to M_\infty (a_D, a)^T$ with the monodromy matrix around infinity given by

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} . \tag{4.36}$$

A bit more abstractly, let $\mathcal{M}'$ be the moduli space with singularities removed. Then, as argued by Seiberg and Witten, the monodromies must form a representation of the fundamental group $\pi_1(\mathcal{M}')$. If $\mathcal{M}$ has only a single puncture at infinity, then the fundamental group is abelian and necessarily so are its representations. In this case, however, a^2 would be a good global coordinate since $M_\infty^2 = \mathbb{1}$, which is in contradiction to what we found in Section 4.1.1.

In order to have non-abelian monodromies, one needs at least two more punctures. Recall from Section 3.3.2 that there is a discrete $\mathbb{Z}_2$ symmetry acting on $u = \frac{1}{2}a^2$ by $u \to -u$. The minimal assumption is hence that there must be two more singularities related by the $\mathbb{Z}_2$ action. As it turns out, this minimal guess leads to very plausible results.

Physically, singularities are caused by additional states becoming massless, which we falsely integrate out in our Wilsonian approach, thereby making $\mathcal{M}$ singular. Classically, this is what one would expect to happen at $u = 0$, where the gauge symmetry is enhanced from $U(1)$ to $SU(2)$, since the corresponding gauge bosons $W^\pm$ do not gain masses through the Higgs mechanism.

Quantum mechanically, this is not what happens. Indeed, Seiberg and Witten used purely physical arguments to show that at one of the singularities a magnetic monopole must become massles. Instead of their insightful physical reasoning [34], there is a simpler, albeit less enlightening mathematical argument, which we give here. Assume that the additional two singularities are at $\pm\Lambda^2$. Then the corresponding monodromy matrices must satisfy

$$M_\infty = M_{\Lambda^2} \cdot M_{-\Lambda^2} \ . \tag{4.37}$$

Even though Eq. (4.37) has infinitely many solutions, they are all related by conjugation and lead to physically equivalent results. We can therefore pick an arbitrary pair and the simplest one to choose is

$$M_{\Lambda^2} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \ . \tag{4.38}$$

To interpret these two matrices physically, let us determine the monodromy around a singularity where a dyon with charge (n_m, n_e) becomes massless, which we denote by $M^{(n_m,n_e)}$. Following [2] we first determine the monodromy around the point where a particle of single electric charge becomes massless. Assuming that this happens at a point $u = u_0$, we know from the BPS bound and Eq. (4.30) that $a(u) \xrightarrow{u \to u_0} 0$. Hence, $a(u)$ is an appropriate local coordinate and can be expanded as

$$a(u) = c(u - u_0) + \dots \tag{4.39}$$

From the β-function of the $U(1)$ theory one further knows that around u_0

$$\tau(a) = -\frac{i}{\pi} \log\left(\frac{a}{\Lambda}\right) \ , \tag{4.40}$$

which gives after integration

$$a_D = -\frac{ia}{\pi} \log\left(\frac{a}{\Lambda}\right) + \frac{ia}{\pi} \ . \tag{4.41}$$

To determine the monodromy, one moves in a loop around u_0 such that $a \to e^{2\pi i} a$ and hence

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_D \\ a \end{pmatrix} = M^{(1,0)} \cdot \begin{pmatrix} a_D \\ a \end{pmatrix} . \tag{4.42}$$

Having found $M^{(1,0)}$, one can use $SL(2,\mathbb{Z})$ duality to find all other monodromies. Assume now that at u_0 a dyon of charge $n = (n_m, n_e)$ becomes massless. To find a dual description in which this dyon appears to have charge $(0,1)$, one uses that under a transformation

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{4.43}$$

the charge transforms as $\hat{n} = n \cdot (S^{-1})$. Here we denote the dual charges $\hat{n}$ by a hat. Since we already know the monodromy $M^{(0,1)}$, we obtain $M^{(n_m,n_e)}$ simply by conjugation:

$$M^{(n_m,n_e)} = S^{-1} M^{(0,1)} S \tag{4.44}$$

Setting $\hat{n} = (0,1)$, one can determine the components of S to be

$$S = \begin{pmatrix} \frac{1+\beta n_m}{n_e} & \beta \\ n_m & n_e \end{pmatrix} , \tag{4.45}$$

which, after insertion into Eq. (4.44), finally yields

$$M^{(n_m,n_e)} = \begin{pmatrix} 1 + 2n_m n_e & 2n_e^2 \\ -2n_m^2 & 1 - 2n_m n_e \end{pmatrix} . \tag{4.46}$$

As a general feature, observe that $\operatorname{Tr} M^{(n_m,n_e)} = 2 \; \forall (n_m, n_e)$. Therefore the $SL(2,\mathbb{Z})$ element corresponding to weak/strong coupling duality $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ does not correspond to a monodromy.

Comparing the general formula to the two monodromy matrices in Eq. (4.38), we find that at $u = \Lambda^2$ a magnetic monopole becomes massless, while the massless state corresponding to the singularity at $u = -\Lambda^2$ is a dyon of charge $(1,-1)$.

4.3 Solution in terms of a curve

Having made all the previous observations about the low-energy effective action of our $\mathcal{N} = 2$ gauge theory in different local patches of the moduli space, the time has come to put all the different pieces together to gain a global picture of the theory. To do so, recall that we viewed (a_D, a) as a section of an $SL(2, \mathbb{Z})$ bundle over the moduli space $\mathcal{M}$ and search for a better geometrical understanding of this structure. We know the monodromies under which (a_D, a) transforms as one circles the punctures of $\mathcal{M}$ and the surviving piece of the $U(1)_R$ symmetry ensures that there is a $\mathbb{Z}_2$ symmetry acting a $u \to -u$. Last of all, we require $\Im \frac{\partial a_D}{\partial a}$ to be positive.

The key insight by Seiberg and Witten was that this moduli space $\mathcal{M}$ can be represented in a form well-known to mathematicians, namely as

$$\mathcal{M} \cong H/\Gamma(2) \tag{4.47}$$

where H is the complex upper half plane and $\Gamma(2) \subset SL(2, \mathbb{Z})$ is the subgroup generated by the monodromy matrices. It is a mathematical fact that the same space describes a certain class of Riemann surfaces of genus one, namely those parametrized as

$$y^2 = (x - \Lambda^2)(x + \Lambda^2)(x - u) \; . \tag{4.48}$$

The moduli space of vacua can therefore equally well be interpreted as being the moduli space of a certain class of curves. Somewhat miraculously, one can exploit this correspondence further and determine all relevant properties of the low-energy effective quantum theory by considering the different geometric properties of the corresponding curve.
First of all, note that y appears quadratically and that the square root has two branches. Hence the total space E_u is a double cover of the complex plane with branch cuts from -1 to 1 and u to ∞ linking the two sheets. In order to see that this corresponds to a genus one surface, consider two homologically equivalent cycles on a torus. Squashing those cycles such that they become lines, one obtains recovers our space with ∞ shifted to some finite value of x. Furthermore, Eq. (4.48) admits a $\mathbb{Z}_4$ symmetry act-

ing as $Z(x, y, u) = (-x, iy, -u)$, of which only a $\mathbb{Z}_2$ subgroup acts non-trivially, since $Z^2(x, -y, u)$ does not have any effect relevant to us, thereby reproducing the $U(1)_R$ remnant of the quantum theory.

Let us now come to the crucial point, namely the geometrical interpretation of the section (a_D, a). Recall our search for a quantity τ satisfying $\Im\tau > 0$. As we will see shortly, this requirement suggests that we pick

$$a = \int_{\gamma_1} \lambda_{\text{SW}} \quad \text{and} \quad a_D = \int_{\gamma_2} \lambda_{\text{SW}} \tag{4.49}$$

for a canonical basis γ_i of the first homology group $H_1(E_u, \mathbb{C})$, i.e. one whose intersection number satisfies

$$\gamma_1 \cdot \gamma_2 = 1 \; . \tag{4.50}$$

From our interpretation of E_u as a torus with two homologically equivalent, but translated cycles reduced to lines, it follows that one can take γ_1 to be the contour surrounding the branch cut from -1 to 1 and γ_2 a contour enclosing the singularities at 1 and u and intersecting the two different branch cuts. Observe now that as u approaches $-1, 1$ or ∞, two branch points collide and cause the curve to become singular.

Furthermore, $\lambda_{\text{SW}} \in H^1(E_u, \mathbb{C})$ must be such that

$$\frac{\mathrm{d}\lambda_{\text{SW}}}{\mathrm{d}u} \sim \omega_1 = \frac{\mathrm{d}x}{y} \; , \tag{4.51}$$

where the proportionality constant is some function independent of x and y and ω_1 is the unique holomorphic differential on our curve E_u. Lastly, λ_{SW}, although allowed to have poles, should not have non-vanishing residues.

Let us now explain the reasons for these choices. Requiring Eq. (4.51) guarantees that

Riemann's second relation holds, namely that

$$\Im\tau = \Im\frac{\mathrm{d}a_D}{\mathrm{d}a} = \Im\frac{\frac{\mathrm{d}a_D}{\mathrm{d}u}}{\frac{\mathrm{d}a}{\mathrm{d}u}} = \Im\frac{\int_{\gamma_1}\omega_1}{\int_{\gamma_2}\omega_1} > 0. \tag{4.52}$$

Therefore every elliptic curve gives rise to an appropriately behaved coupling constant. As noted by [34], the converse case is also true: Given τ with $\Im\tau > 0\ \forall u$, then the transformation of τ and (a_D, a) under monodromies around singularities uniquely determines the family of curves, leading once again to the conclusion that there is a $1:1$ relation between the low-energy effective theory for some u and a curve as in Eq. (4.48) with the same parameter.

In order to determine the proportionality constant of Eq. (4.51), one uses the known behaviour of (a_D, a) around one of the singularities, which fixes it uniquely. It then remains to be shown that (a_D, a) behave as expected around the other singularities as well. The correct choice turns out to be

$$\lambda_{\mathrm{SW}} = \frac{\omega_2 - u\omega_1}{\sqrt{2}\pi} = \frac{\mathrm{d}x}{\sqrt{2}\pi y}(x-u)\ . \tag{4.53}$$

Take γ_i as above, then

$$a = \frac{\sqrt{2}}{\pi}\int_{-\Lambda^2}^{\Lambda^2}\mathrm{d}x\frac{\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}} \xrightarrow{u\to\infty} \frac{\sqrt{2u}}{\pi}\int_{-\Lambda^2}^{\Lambda^2}\frac{\mathrm{d}x}{\sqrt{\Lambda^4-x^2}} = \sqrt{2u} \tag{4.54}$$

and

$$\begin{aligned} a_D &= \frac{\sqrt{2}}{\pi}\int_{\Lambda^2}^{u}\mathrm{d}x\frac{\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}} = \frac{\sqrt{2}}{\pi}\int_{\Lambda^2/u}^{1}\mathrm{d}z\,u\frac{\sqrt{uz-u}}{\sqrt{u^2z^2-\Lambda^4}} \\ &= \frac{\sqrt{2u}}{\pi}\int_{\Lambda^2/u}^{1}\frac{\sqrt{z-1}}{\sqrt{z^2-\Lambda^4/u^2}} \xrightarrow{u\to\infty} \sqrt{2u}\frac{i}{\pi}\log\left(\frac{u}{\Lambda^2}\right)\ . \end{aligned} \tag{4.55}$$

Analogously, for $u \to 1$ one finds that

$$a = \frac{4\Lambda}{\pi} - \frac{(u-\Lambda^2)\log(u-\Lambda^2)}{2\pi} \quad \text{and} \quad a_D = \frac{i}{2\Lambda}(u-\Lambda^2)\ , \tag{4.56}$$

thereby reproducing the monodromies expected from Eq. (4.38).

In fact, one can evaluate both integrals in terms of hypergeometric functions, namely

$$a = \sqrt{2(u+1)}F\left(-\frac{1}{2},\frac{1}{2},1;\frac{2}{u/\Lambda^2+1}\right) \tag{4.57}$$

$$a_D = \frac{i}{2\Lambda}(u-\Lambda^2)F\left(\frac{1}{2},\frac{1}{2},2;\frac{1-u/\Lambda^2}{2}\right) , \tag{4.58}$$

whose inversion gives $u(a)$, $a_D(a)$ and $\mathcal{F}(a)$. From the last term, one can then easily read off the coefficients of the instanton contributions.

Having found explicit expressions for $a(u)$ and $a_D(u)$, let us summarize the findings by Seiberg and Witten: The low-energy effective action of $\mathcal{N}=2$ Super-Yangs-Mills theory with gauge group $SU(2)$ is a $U(1)$ theory, whose prepotential $\mathcal{F}$ is severely constrained by supersymmetry. $\mathcal{F}$ further exhibits an $SL(2,\mathbb{Z})$ duality relating infinitely many different descriptions of the same physical theory by transformations on the fields (a_D, a). The moduli space $\mathcal{M}$ of the theory is shown to be the same as the moduli space of a certain class of genus one Riemann surfaces and many geometric properties can be interpreted as gauge theory features. Among these are the variables (a_D, a), which are determined by calculating the integral of a (up to exact forms) uniquely defined *Seiberg-Witten* differential λ_{SW} over a basis of 1-cycles of the Riemann surface. Having found (a_D, a) the coupling constant τ can be determined.

Physically, singularities of the curves, i.e. points in the moduli space of the Seiberg-Witten curve at which some cycle degenerates, correspond to additional particles becoming massless. Unlike in the classical case, there is no point in the moduli space at which gauge bosons become massless and the full gauge symmetry is restored. Instead, monopoles or dyons have zero mass at the singularities of $\mathcal{M}$ corresponding to $u \to \pm\Lambda^2$. Remarkly, the whole analysis relied mainly on global properties of the moduli space and used almost no information about the instanton solutions or even their moduli spaces themselves.

5 Reproducing Seiberg-Witten from field theory

After the unexpected solution by Seiberg and Witten, interest increased in verifying their results independently by determining the instanton contributions using a more conventional field theoretic approach. Due to the technical difficulty of the computation, it took eight years and a series of works until Nekrasov succeeded in doing so.
In his calculation, Nekrasov uses a variety of fairly advanced techniques from pure mathematics and topological field theory in addition to the instanton techniques we have studied. Due to the constraints of time and space, we only try to point out the gist of his calculation, hoping to make the concepts behind it somewhat accessible. To introduce the necessary background, this chapter first summarizes the key points of cohomological field theory and presents Witten's twisted $\mathcal{N} = 2$ field theory, which is an interesting subject in its own right. Readers interested in more detail can consult the reviews [9, 11]. Afterwards, we outline the concept of Nekrasov's own work, namely [32] and references therein.

5.1 Cohomological field theory

Let us now explain in crude terms what is required for a quantum field theory in order to be a cohomological field theory by roughly following the exposition in [9].
Given a field theory on some arbitrary manifold $\mathcal{M}$, we call it *cohomological* if there is a fermionic operator $\mathcal{Q}$ whose action on some functional V gives the action of the theory

$$S = \{\mathcal{Q}, V\} \ . \tag{5.1}$$

Furthermore, $\mathcal{Q}$ is required to be nilpotent and the path integral measure invariant under its action. $\mathcal{Q}$ therefore generates a symmetry of the theory, since

$$\delta S = \{\mathcal{Q}, S\} = 0 \ . \tag{5.2}$$

Since the vacuum should be invariant under any unbroken symmetry, it satisfies

$$\mathcal{Q} \left|0\right\rangle = 0 \ . \tag{5.3}$$

To see why these requirements have rather drastic implications for the theory, begin by noting that any operator $\mathcal{O}$, which can be expressed as $\mathcal{O} = \{\mathcal{Q}, \chi\}$ for some χ, has zero expectation value:

$$\langle \mathcal{O} \rangle = \langle 0 | \{\mathcal{Q}, \chi\} | 0 \rangle = \langle 0 | (\mathcal{Q}\chi + \chi\mathcal{Q}) | 0 \rangle = 0 \; . \tag{5.4}$$

Consider now the partition function of the theory

$$Z = \int [\mathrm{d}\Phi] \, \mathrm{e}^{-S[\Phi, g]} \; , \tag{5.5}$$

where Φ denotes the set of all fields and g is the metric on $\mathcal{M}$. Varying g infinitesimally and assuming that the path integral measure is independent of g, one obtains

$$\begin{aligned} \delta_g Z &= -\int [\mathrm{d}\Phi] \, \mathrm{e}^{-S[\Phi, g]} \, \delta_g S \\ &= -\int [\mathrm{d}\Phi] \, \mathrm{e}^{-S[\Phi, g]} \, \{\mathcal{Q}, \frac{\delta V}{\delta g^{\alpha\beta}} \delta g^{\alpha\beta}\} \\ &= \langle \{\mathcal{Q}, \frac{\delta V}{\delta g^{\alpha\beta}} \delta g^{\alpha\beta}\} \rangle \\ &= 0 \; . \end{aligned} \tag{5.6}$$

Hence, the partition function is independent of the metric on $\mathcal{M}$ and must therefore be a topological invariant of the underlying manifold. For that reason, cohomological field theories are a special class of topological field theories and were first examined by Witten [44]. To see why they deserve their name, let us ask whether there are additional topological invariants defined by the expectation value of some operator $\mathcal{O}$. Varying the metric, one finds

$$\begin{aligned} \delta_g \langle \mathcal{O} \rangle &= \int [\mathrm{d}\Phi] \, \mathrm{e}^{-S[\Phi, g]} \, (\delta_g \mathcal{O} - \delta_g S \cdot \mathcal{O}) \\ &= \langle (\delta_g \mathcal{O} - \delta_g S \cdot \mathcal{O}) \rangle \; . \end{aligned} \tag{5.7}$$

If $\mathcal{O}$ further fulfills

$$\{\mathcal{Q}, \mathcal{O}\} = 0 \qquad \text{and} \qquad \delta_g \mathcal{O} = \{\mathcal{Q}, R\} \tag{5.8}$$

for some operator R, then

$$\delta_g \langle \mathcal{O} \rangle = \langle \{ \mathcal{Q}, R + V \cdot \mathcal{O} \} \rangle = 0 \tag{5.9}$$

and $\langle \mathcal{O} \rangle$ is indeed another metric independent quantity. On the other hand, we saw in Eq. (5.4) that operators that can be expressed as $\mathcal{Q}$ acting on some other operator, the $\mathcal{Q}$-*exact* operators, give trivial observables. We are therefore interested in operators $\mathcal{O}$ which are annihilated by $\mathcal{Q}$, so-called $\mathcal{Q}$-*closed* operators, which are not $\mathcal{Q}$-exact. Mathematically speaking, we are studying the cohomology of $\mathcal{Q}$: Two topological invariants are the same if their corresponding operators belong to the same cohomology class, i.e. they differ only by a $\mathcal{Q}$-exact term.

Last but not least, let us point out another crucial property of cohomological field theories, namely the fact that their semi-classical approximation is exact. To see how this feature emerges, rescale the action by a dimensionless parameter t. Consider now the variation with respect to t:

$$\begin{aligned} \delta_t Z(t) &= \delta_t \int [\mathrm{d}\Phi]\, \mathrm{e}^{-t \cdot S[\phi, g]} \\ &= \langle \{ \mathcal{Q}, V \} \rangle \delta t \\ &= 0 \end{aligned} \tag{5.10}$$

Since $Z(t)$ is independent of t, one can evaluate the path integral limit $t \to \infty$, where $Z(t)$ *localizes* on the minima of the action, which correspond to the classical solutions.

5.2 Twisted $\mathcal{N} = 2$ theory

Heaving specified what is meant by a cohomological field theory, let us now present what is possibly the most famous example, the so-called *Donaldson-Witten-theory.* First written down by Witten [45], its action reads as follows:

$$S = \int_{\mathcal{M}} \sqrt{g}\, \mathrm{d}^4x \ \mathrm{Tr}\Bigg(\frac{1}{4} F_{mn}F^{mn} + \frac{1}{2}\phi^\dagger D_m D^m \phi - i\eta D_m \psi^m + i D_m \psi_n \chi^{+mn} \\ - \frac{i}{8}\phi[\chi^+_{mn}, \chi^{+mn}] - \frac{i}{2}\phi^\dagger[\psi_m, \psi^m] - \frac{i}{2}\phi[\eta,\eta] - \frac{1}{8}[\phi,\phi^\dagger]^2 + \frac{1}{4\sqrt{g}} F_{mn}\tilde{F}^{mn} \Bigg) \quad (5.11)$$

In this expression all fields transform in the adjoint representation of the gauge group and at least the bosonic fields look fairly ordinary: ϕ is a complex scalar field and F_{mn} is the usual field strength tensor. On the other hand, the fermionic fields η, ψ_m and χ^+_{mn} transform in the scalar, vector and self-dual second antisymmetric representation respectively.

As it is a cohomological field theory, its action can be expressed as $S = \{\mathcal{Q}, V\}$, where

$$V = \mathrm{Tr}\left(\frac{1}{4} F^{mn}\chi^+_{mn} + \frac{1}{2}\psi_m D^m \phi^\dagger - \frac{1}{4}\eta[\phi,\phi^\dagger] \right) \quad (5.12)$$

and $\mathcal{Q}$ is a nilpotent operator that we will specify shortly.

Despite its unconventional appearance, this action is related to the well-known $\mathcal{N} = 2$ pure Super-Yang-Mills theory by a pure change of notation and the inclusion of the topological term $F_{mn}\tilde{F}^{mn}$. Let us now see how this arises.

From the previous chapter, we are familiar with both the field content and the global symmetry group of $\mathcal{N} = 2$ SYM. Dropping the anomalous $U(1)$ part of the R-symmetry, the symmetry group is $SU(2)_L \times SU(2)_R \times SU(2)_I$. Consider now to replace $SU(2)_R \times SU(2)_I$ by its diagonal subgroup, which we call $SU(2)_{R'}$. Under the new "Lorentz group" the bosonic fields retain their transformation behavior, but the fermions transform under different representations, since the two spin 1/2 fields split up into a scalar, a vector and a self-dual part, as can be seen from Table 6. Since the supercharges of the original theory split analogously, one can identify the operator $\mathcal{Q}$ with the scalar supercharge part. We therefore see that the Donaldson-Witten action can be obtained from ordinary $\mathcal{N} = 2$ SYM theory by a simple twist and it is therefore sometimes called *twisted* $\mathcal{N} = 2$ theory.

Old field	$SU(2)_L$	$SU(2)_R$	$SU(2)_I$	$SU(2)_L \times SU(2)_{R'}$	New field(s)
A_m	[1]	[1]	[0]	[1; 1]	A_m
ϕ	[0]	[0]	[0]	[0; 0]	ϕ
ψ_α	[1]	[0]	[1]	[1; 1]	ψ_m
$\lambda_{\dot\alpha}$	[0]	[1]	[1]	[0; 0] + [0; 2]	$\eta + \chi^+_{mn}$

Table 6: Pure $\mathcal{N} = 2$ field content in untwisted and twisted notation

By varying the action one finds that the classical equations of motion are

$$F_{mn} = -\tilde{F}_{mn} \tag{5.13}$$

and

$$D_m\psi_n - D_n\psi_m + \epsilon_{mnrs}D^r\psi^s = 0 \tag{5.14}$$

$$D_m\psi^m = 0 \tag{5.15}$$

for the fermions. Recalling the results from Chapter 2 we hence see that the path integral reduces to an integral over the anti-instanton moduli space.

Having defined the action, we know that the first non-trivial topological invariant is the partition function of the theory. On the other hand, it remains to be seen which other invariants there are. As discussed in the previous section, we are looking for representatives of non-trivial cohomology classes of $\mathcal{Q}$ which transform by $\mathcal{Q}$-exact terms under variations of the metric.

It turns out that choosing gauge invariant polynomials of ϕ gives such operators, since ϕ does not depend on g and is invariant under $\mathcal{Q}$. Nevertheless, one can argue that it is not $\mathcal{Q}$-exact and it is therefore a suitable choice. Since gauge invariant polynomials are in one to one correspondence with Casimir operators of the respective gauge group, there are exactly rk(G) independent polynomials. In the simplest case of $SU(2)$ there is only the one known from the previous chapter and we define

$$\mathcal{O}^{(0)}(x) = \frac{1}{2}\,\mathrm{Tr}\,\phi^2(x)\ . \tag{5.16}$$

Given $\mathcal{O}^{(0)}$ and a choice of k points $\{x_1, \ldots, x_k\}$ on $\mathcal{M}$, we can then define the following topological invariant

$$Z(k) = \langle \mathcal{O}^{(0)}(x_1) \dots \mathcal{O}^{(0)}(x_k) \rangle \ . \tag{5.17}$$

Since $Z(k)$ is metric independent, it cannot depend on the choice of points. This observation led Witten to find that

$$\mathrm{d}_{x_1} Z(k) = \langle \mathrm{d}\mathcal{O}^{(0)}(x_1) \dots \mathcal{O}^{(0)}(x_k) = 0 \tag{5.18}$$

implies that

$$\mathrm{d}\mathcal{O}^{(0)}(x) = i\{\mathcal{Q}, \mathcal{O}^{(1)}(x)\} \tag{5.19}$$

for some one-form $\mathcal{O}^{(1)}(x)$. In fact, this process can be continued recursively up to $\mathcal{O}^{(4)}(x)$ by solving

$$\mathrm{d}\mathcal{O}^{(k)}(x) = i\{\mathcal{Q}, \mathcal{O}^{(k+1)}(x)\} \tag{5.20}$$

for $k \in \{0, 1, 2, 3\}$. One finds

$$\begin{aligned} \mathcal{O}^{(1)} &= \mathrm{Tr}\left(\phi \wedge \psi\right) & \mathcal{O}^{(2)} &= \mathrm{Tr}\left(\frac{1}{2}\psi \wedge \psi + i\phi \wedge F\right) \\ \mathcal{O}^{(3)} &= i\,\mathrm{Tr}\left(\psi \wedge F\right) & \mathcal{O}^{(4)} &= -\frac{1}{2}\,\mathrm{Tr}\left(F \wedge F\right) \ , \end{aligned} \tag{5.21}$$

where ψ and F are taken to be 1- and 2-forms. Using these new forms, one can define additional operators as follows: Take γ to be a homology cycle of dimension k and let

$$I(\gamma) = \int_\gamma \mathcal{O}^{(k)} \ , \tag{5.22}$$

then this defines another $\mathcal{Q}$-closed operator:

$$i\{\mathcal{Q}, I(\gamma)\} = \int_\gamma i\{\mathcal{Q}, \mathcal{O}^{(k)}\} = \int_\gamma \mathrm{d}\mathcal{O}^{(k-1)} = \int_{\partial\gamma} \mathcal{O}^{(k-1)} = 0 \tag{5.23}$$

Furthermore, if γ is a homology boundary, i.e. $\gamma = \partial\beta$ for some β, then I is $\mathcal{Q}$-exact:

$$I(\gamma) = \int_{\partial\beta} \mathcal{O}^{(k)} = \int_{\beta} \mathrm{d}\mathcal{O}^{(k)} = i\{\mathcal{Q}, \int_{\beta} \mathcal{O}^{(k-1)}\} \tag{5.24}$$

Topological invariants associated with $I(\gamma)$ therefore only depend on the homology class of γ and we can define our most general invariant:

$$Z(\gamma_1, \ldots, \gamma_r) = \langle I(\gamma_1) \ldots I(\gamma_r) \rangle \tag{5.25}$$

The significance of this field theory and its observables lies in the connection to Donaldson theory. One of Donaldson's great achievements was his discovery of certain topological invariants of four-manifolds, which he found by studying certain instanton solutions on those manifolds. Witten showed that these invariants are the same as those arising from the twisted $\mathcal{N} = 2$ SYM-theory.

Due to the connection to pure mathematics, there are many more things one could say about Donaldson-Witten theory and we refer to [9] for more details and further references.

5.3 Calculating the full prepotential from field theory

Having discovered that $\mathcal{N} = 2$ gauge theory can be twisted such that various observables become topological invariants, one could hope to perform the localized integral. In fact, Hollowood used the twisted supercharge to show [22] that the integral of Eq. (3.36) localizes on the minima of the effective instanton action, i.e. those points at which all instantons shrink to zero size. Nevertheless, there remains a variety of problems, since the resulting spaces are both non-compact and singular. Although these problem can be overcome using the resolved instanton moduli spaces mentioned in Section 2.4.2, the resulting integrals remain difficult and were only evaluated for the case of one and two instantons [23].

In this last section, we sketch Nekrasov's solution of the problem. The additional ingredient of his computation is to use more symmetries of the moduli space $\mathfrak{M}_k$. As we noted in Chapter 2, instantons with gauge group G have a natural action of $G \times SO(4)$ on their moduli space, where G corresponds to global gauge transformations and $SO(4)$

is the usual group of four-dimensional rotations. Instead of examining $\mathfrak{M}_k$, Nekrasov instead decided to study the cohomology of $\mathfrak{M}_k/(G \times \mathbf{T}^2)$, where $\mathbf{T}^2$ is the maximal torus of $SO(4)$. The proper framework of studying the cohomology of this singular space is called *equivariant cohomology* and one denotes the respective cohomology groups by $H^*_{G\times\mathbf{T}^2}(\mathfrak{M}_k)$.

In the following, we will completely neglect all issues arising due to zero size instantons and assume implicitly that we are considering the smoothly resolved instanton moduli spaces instead (see [32] for a justification).

5.3.1 Defining UV and IR observables

Without going into more detail on the mathematical side, let us show how the invariants of the conventional twisted $\mathcal{N} = 2$ theory are modified:

As before, the field content is given in Table 6, but let us pay a bit more respect to the supercharges and superspace this time. Since they transform in the same representations as the fermions, they split up in the same way and one obtains the twisted supercharges and superspace coordinates

$$\mathcal{Q}, \mathcal{Q}^+_{mn}, G_m \qquad \text{and} \qquad \theta^m, \bar{\theta}, \bar{\theta}^{+mn} \ . \tag{5.26}$$

In terms of the twisted superspace, a superfield can be expanded as

$$\varPhi = \phi + \theta^m \psi_m + \frac{1}{2} F_{mn} \theta^m \theta^n + \dots \ , \tag{5.27}$$

or, using form notation:

$$\varPhi = \phi + \psi + F + \dots \tag{5.28}$$

Studying equivariant cohomology means that our nilpotent operator becomes

$$\tilde{\mathcal{Q}} = \mathcal{Q} + E_a \varOmega^a_{mn} x^n G^m \ , \tag{5.29}$$

where $\varOmega^a_{mn} x^n \partial^m$, $a = 1, \dots, 6$ are the vector fields generating rotations of $SO(4)$ and $E_a \in \mathfrak{so}(4)$. Given $\tilde{\mathcal{Q}}$, one can proceed in the spirit of cohomological field theory and construct $\tilde{Q}$-closed operators:

$$\mathcal{O}_P^{\Omega(E)} = \int_{\mathbb{R}^4} \Omega(E) \wedge P(\Phi) \tag{5.30}$$

There are several comments to be made about this formula:

1. $\Omega(E)$ is an inhomogeneous form on $\mathbb{R}^4$ that must transform *equivariantly* under $SO(4)$, i.e.

$$g^* \Omega(E) = \Omega(g^{-1} E g) \qquad \forall\, g \in SO(4)\ . \tag{5.31}$$

It is further required to be closed under the equivariant differential

$$D\Omega(E) = (\mathrm{d} + \iota_{V(E)})\Omega(E) = 0\ , \tag{5.32}$$

where $V(E)$ is the vector field $E_a \Omega^a_{mn} x^n \partial^m$ generating rotations on $\mathbb{R}^4$.

2. $P(\Phi)$ must be a gauge invariant polynomial and all terms in the integrand of Eq. (5.30) are understood to be dropped if their total degree is not equal to four.

Next of all, let us find a form $\Omega(E)$ satisfying the above conditions. As discussed in [32], it turns out that requiring $SO(4)$ invariance is too strong a restriction: There are no non-constant forms. One therefore breaks the $SO(4)$ symmetry down to $U(2)$ by choosing a symplectic structure on $\mathbb{R}^4$:

$$\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4 \tag{5.33}$$

At the same time, ω defines a complex structure on $\mathbb{R}^4$, turning it into $\mathbb{C}^2$ by assigning complex coordinates $z_1 = x^1 + ix^2$ and $z_2 = x^3 + ix^4$. To construct an equivariantly closed form, one can now exploit the standard fact that the symplectic form defines a moment map $\mu(\mathbb{C}^2) \to \mathfrak{u}(2)^*$ via

$$\mathrm{d}\mu(E) = \iota_{V(E)} \omega\ . \tag{5.34}$$

Parametrizing the maximal torus of $SO(2) \times SO(2) \subset SO(4)$ by $E = (\epsilon_1, \epsilon_2)$, the moment map is given as

$$H(E) = \mu(E) = \epsilon_1 |z_1|^2 + \epsilon_2 |z_2|^2 \; . \tag{5.35}$$

From Eq. (5.34) and $\iota_{V(E)} H(E) = \mathrm{d}\omega = 0$ it follows that the inhomogeneous form

$$\Omega(E) = \omega - H(E) \tag{5.36}$$

is indeed closed under D.

Let us choose $P(\Phi) = \mathrm{Tr}(\Phi^2)$ and define

$$\begin{aligned} Z(a, \epsilon_1, \epsilon_2) &= \left\langle \exp\left(-\frac{1}{4\pi^2} \int_{\mathbb{R}^4} (\omega - H) \wedge \mathrm{Tr}(\Phi^2) \right) \right\rangle_a \\ &= \left\langle \exp\left(-\frac{1}{4\pi^2} \int_{\mathbb{R}^4} \omega \wedge \mathrm{Tr}\left(\phi \wedge F + \frac{1}{2} \psi \wedge \psi \right) - H \wedge \mathrm{Tr}\left(F \wedge F \right) \right) \right\rangle_a \end{aligned} \tag{5.37}$$

Here a is the expectation value of the scalar field which takes value in the Cartan subalgebra of $\mathfrak{g}$ and the path integral is to be evaluated in a background such that $\langle \phi \rangle = a$. Restricting to $G = SU(2)$, it can hence be identified with the variable of Seiberg and Witten denoted by the same letter.

Eq. (5.37) is precisely the topological invariant we are after and one can show [32] that it has in fact trivial cohomology. Its crucial advantage is that it also has a nice interpretation in the strongly coupled IR regime, which we will now explain.

In fact, if H were constant, then its insertion in Eq. (5.37) would correspond only to rescaling the effective coupling constant of the theory by a factor of e^{-H}. Since it varies in reality, this picture is correct only up to derivatives of H. Additionally, ω must be taken into account, too. Schematically, they renormalize the energy scale of the theory as

$$\Lambda \to \Lambda \cdot \mathrm{e}^{-H+\omega} \; , \tag{5.38}$$

where ω is really a function on the twisted superspace. Inserting this renormalization into the prepotential of the low-energy effective theory and expanding one finds

$$\mathcal{F}(a; \Lambda \mathrm{e}^{-H+\omega}) = \mathcal{F}(a; \Lambda \mathrm{e}^{-H}) + \omega \frac{\partial \mathcal{F}(a; \Lambda \mathrm{e}^{-H})}{\partial \log(\Lambda)} + \frac{1}{2} \omega \wedge \omega \frac{\partial^2 \mathcal{F}(a; \Lambda \mathrm{e}^{-H})}{\partial \log(\Lambda)^2} \ . \tag{5.39}$$

Of these terms, all but the last one become coupled to the gauge field after integrating over superspace. Going to long distances, they become suppressed. Using that derivatives of H are proportional to ϵ_i and evaluating Eq. (5.37) in the far IR, one therefore finds that

$$Z(a; \epsilon_1, \epsilon_2) = \exp\left(-\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \omega \wedge \omega \frac{\partial^2 \mathcal{F}(a; \Lambda \mathrm{e}^{-H})}{\partial \log(\Lambda)^2} \right) + \mathcal{O}(\epsilon_i) \ , \tag{5.40}$$

since all other terms vanish. Now one can insert the expressions for ω and H, perform the integral over $\mathbb{R}^4$ and compare with Eq. (4.11) to find:

$$Z(a; \epsilon_1, \epsilon_2) = \exp\left(\frac{\mathcal{F}_{\text{inst}}(a; \Lambda)}{\epsilon_1 \epsilon_2} + \mathcal{O}(1) \right) \tag{5.41}$$

5.3.2 Integrating over the instanton moduli space

Having found the relation between $Z(a; \epsilon_1, \epsilon_2)$ and the non-perturbative part of the prepotential, what remains to be done is the integration over the instanton moduli space, since we saw in the previous subsection that this is the locus on which the path integral localizes. This can be achieved in various ways, but for instanton numbers larger than two, they all rely on localization principles and are fairly technical. One method to perform the integration is to make use of the ADHM construction presented in Chapter 2 and its stringy interpretation as the Higgs branch of the moduli space of a certain $\mathcal{N} = 2$ gauge theory. As we discussed, the resulting instanton moduli space is a Hyperkähler quotient with moment maps given by Eqs. (2.35) and (2.36). In order to determine the equivariant volume of such spaces, Moore, Nekrasov and Shatashvili developed integration techniques and actually computed the equivariant volume of the instanton moduli spaces [28] five years before Nekrasov's paper.

Instead of repeating their derivation, we only present their result:

$$Z_k(a;\epsilon_1,\epsilon_2) = \frac{1}{k!}\frac{(\epsilon_1+\epsilon_2)^k}{\epsilon_1^k\epsilon_2^k}\oint\prod_{i=1}^{k}\frac{\mathrm{d}\phi_i}{2\pi i}\prod_{j=1}^{k}\frac{1}{(\phi_j^2-a^2)((\phi_j+\epsilon_1+\epsilon_2)^2-a^2)} \times\prod_{m\neq n}^{k}\frac{\phi_{mn}(\phi_{mn}+\epsilon_1+\epsilon_2)}{(\phi_{mn}+\epsilon_1)(\phi_{mn}+\epsilon_2)} \tag{5.42}$$

Here $\phi_{ij} = \phi_i - \phi_j$ and we split the partition function $Z(a;\epsilon_1,\epsilon_2)$ into the different contributions of definite instanton charge:

$$Z(a;\epsilon_1,\epsilon_2) = \sum_{k=1}^{\infty}\Lambda^{4k}Z_k(a;\epsilon_1,\epsilon_2) \tag{5.43}$$

Note that the integral in Eq. (5.42) is to be understood as a contour integral, picking up residues at the poles of the integrand.

Despite the fact that the integral can be evaluated in terms of a sum over Young tableaux (see [28, 32]), direct comparison with the results by Seiberg and Witten still remains a challenge. In a tour de force, Nekrasov determined the three instanton contribution for any gauge group $SU(N)$ as well as the five instanton contribution for $SU(2)$ and $SU(3)$ and found perfect agreement.

6 Summary

The main objective of this dissertation was to study instanton solutions in quantum field theory and the connection to other topics arising in this context. In doing so, one is quickly led to the rich geometrical structure of the instanton moduli space and, via the stringy interpretation of the ADHM construction, a helpful connection to string theory. Even though the focus of this thesis was put on studying instantons for their own sake, we noted in Chapter 3 that they do appear in a variety of physical applications. Despite the fact that instanton effect can be difficult to compute quantitatively, we found that their inclusion improves the qualitative understanding of quantum field theory.

Nevertheless, as is frequently the case, exact statements about instantons can be made in the much more restricted cases of supersymmetric gauge theories and Chapters 4 and 5 solely dealt with pure $\mathcal{N} = 2$ SYM theory. Again, we encountered a plethora of highly non-trivial mathematical structures that, at least naively, one might not have expected, such as for example the emergence of a class of Riemann surfaces.

As we have pointed out before, this review is incomplete in a variety of ways. While instantons and their moduli space have been under study for several decades by now, there is still much to be understood. As we illustrated in Chapter 2, there exists a rich interplay between supersymmetry and instanton moduli spaces. Over the past two decades, much has been learned about supersymmetric gauge theory and one can therefore hope that techniques that are newly discovered in the their study may shed more light on instantons as well.

Before closing, let us point out that there are many directions left unexplored by this thesis. Among other things, we omitted the generalization of Seiberg and Witten's discussion to other gauge groups and the inclusion of matter and the corresponding work by Nekrasov. Maybe more importantly, there is a beautiful geometrical interpretation of the Seiberg-Witten curve in terms of M-theory [25, 46] and a number of recent developments [17] that would have been interesting to examine more closely.

Nevertheless, we hope to have given a somewhat instructive introduction to the fascinating topic of instantons that can serve as a starting point for reading some of the more advanced literature.

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